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Analytic Theory of L-Functions: Explicit Formulae, Gaps Between Zeros and Generative Computational Methods

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Zürich, 2016

Declaration of Authorship

I, Patrick KÜHN, declare that this thesis titled, 'Analytic Theory of L-Functions: Explicit Formulae, Gaps Between Zeros and Generative Computational Methods' and the work presented in it are my own. I confirm that:

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"The art of doing mathematics consists in finding that special case which contains all the germs of generality."

David Hilbert

UNIVERSITÄT ZÜRICH

Abstract

Mathematisch-naturwissenschaftlichen Fakultät
Institut für Mathematik

Doctor of Philosophy

Analytic Theory of L-Functions: Explicit Formulae, Gaps Between Zeros and Generative Computational Methods

by Patrick KÜHN

Several aspects of the analytic and computational theory of L-functions are covered in this thesis. These include:

1. explicit formulae involving the Cohen-Ramanujan sum and the Möbius function are proved using analytic methods;
2. the largest gap between zeros of any entire L-function of any degree is improved to 41.54 under the assumption of the Grand Riemann hypothesis and the Ramanujan hypothesis;
3. generative computational methods for finding Dirichlet coefficients of self-dual L-functions are introduced;
4. the mollification of $\zeta(s) + \zeta'(s)$ put forward by Feng is computed by analytic methods, clarifying the current situation on the percentage of non-trivial zeros of the Riemann zeta-function on the critical line.

In meiner Doktorarbeit werden verschiedene Aspekte der analytischen und rechnerischen Theorie der L-Funktionen betrachtet. Diese beinhalten:

1. explizite Formeln für die Summe von Cohen-Ramanujan und für die Möbius-Funktion werden mit analytischen Methoden bewiesen;
2. der grösste Intervall zwischen zwei Nullstellen jeder ganzen L-Funktion beliebigen Grades wird unter der Annahme der verallgemeinerten Riemannschen Vermutung und der Ramanujan-Vermutung auf 41.54 verkleinert;
3. rechnerische Erzeugungsmethode für die Entdeckung der Dirichlet Koeffizienten von selbst-dualer L-Funktionen werden vorgestellt;
4. die von Feng vorgestellte Glättung von $\zeta(s) + \zeta'(s)$ wird mit Hilfe analytischer Methoden berechnet. Dies verdeutlicht die aktuelle Situation über den nichttrivialen Prozentsatz der Nullstellen der Riemannschen Zeta-Funktion entlang der kritischen Geraden.

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Symbols

$\mathbb{H}$	The upper half-plane $\{x+iy \ : \ y>0\}$
$\Delta_{\mathbb{H}}$	The hyperbolic Laplacian differential operator, defined as $\Delta_{\mathbb{H}} = y^2\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)$
$SL(n,\mathbb{Z})$	The special linear group of dimension n, i.e. the group under matrix multiplication of $n\times n$ matrices in $\mathbb{Z}$ with determinant equal to 1
Γ	A congruence subgroup, i.e. a subgroup of $SL(2,\mathbb{Z})$ containing the kernel of the canonical map $SL(2,\mathbb{Z})\to SL(2,\mathbb{Z}/N\mathbb{Z})$ for a certain level N
$M_k(\Gamma)$	The space of modular forms of weight k for the congruence group Γ
$S_k(\Gamma)$	The space of cusp forms of weight k for the congruence group Γ
T_n	The Hecke operators on Γ for each $n\geq 1$
$\mathcal{S}$	The Selberg class of L-functions
$\mathcal{S}_{hol}$	The subclass of the Selberg class consisting of entire L-functions
$h,j,k,l,m,n,\ldots$	Natural numbers
$p,p_1,p_2,q,\ldots$	Prime numbers
$A,B,C,C_1,C_2,\ldots$	Absolute constants (not necessarily the same at each occurrence in a proof)
$\delta,\delta_0,\varepsilon,\varepsilon_0,\ldots$	Arbitrarily small positive constants (not necessarily the same at each occurrence in a proof)
$t,x,y,\ldots$	General real variables
$s=\sigma+it,u,w,z,\ldots$	General complex variables, $\sigma=\mathrm{Re}(s)$ and $t=\mathrm{Im}(s)$
$\mathrm{Res}_{s=s_0}\,f(s)$	The residue at s_0 of the meromorphic function $f(s)$
$\Gamma(s)$	The Gamma function $\Gamma(s)=\int_0^\infty e^{-t}t^{s-1}dt$
$\Gamma(s,x)$	The upper incomplete Gamma function $\Gamma(s,x)=\int_x^\infty e^{-t}t^{s-1}dt$
γ	The Euler-Mascheroni constant $\gamma=-\Gamma'(1)$
$\Gamma_{\mathbb{R}}(s)$	The Gamma-$\mathbb{R}$ function $\Gamma_{\mathbb{R}}(s)=\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)$
$\Gamma_{\mathbb{C}}(s)$	The Gamma-$\mathbb{C}$ function $\Gamma_{\mathbb{C}}(s)=2(2\pi)^{-s}\Gamma\left(s\right)$
$\zeta(s)$	The Riemann zeta-function
$\Lambda_\zeta(s)$	The completed Riemann zeta-function $\Lambda_\zeta(s)=\Gamma_{\mathbb{R}}(s)\zeta(s)$

$\xi(s)$	The Riemann ξ-function $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s)$
$\rho = \beta + i\gamma$	A nontrivial zero of the Riemann zeta-function
$\rho_\chi = \beta + i\gamma$	A nontrivial zero of $L(s,\chi)$
$\chi(n)$	A Dirichlet character modulo q, i.e. a homomorphism $\chi : (\mathbb{Z}/q\mathbb{Z})^* \to \mathbb{C}^*$
$L(s,\chi)$	A Dirichlet L-function associated to a primitive Dirichlet character χ mod q
$\tau(\chi)$	The Gauss sum of a Dirichlet character χ
$L(s,f)$	An L-function associated to an automorphic form f (a classical modular form, a Maass cusp form, ...)
$\zeta_K(s)$	The Dedekind zeta-function associated to the number field K
$L(s,E)$	The Hasse-Weil L-function associated to an elliptic curve E over a number field
$L(s)$	A general L-function of degree $d \geq 1$ in the Selberg class $\mathcal{S}$
$b(n)$	The Dirichlet coefficients of a general L-function
$\Lambda(s) = \Lambda_L(s)$	The completed L-function satisfying a functional equation
$\overline{F}(s)$	The conjugate of a complex function, defined as $\overline{F}(s) = \overline{F(\overline{s})}$
$H_L(x)$	The H-invariants of an L-function defined in Chapter 1
$d = d_L$	The degree of an L-function
$N = N_L$	The conductor or level of the L-function
$\varepsilon = \varepsilon_L$	The sign or root number of the L-function, with $\lvert\varepsilon\rvert = 1$
$L_p(x)$	The local factor at a good prime p of the L-function in its Euler product
$\tilde{L}_p(x)$	The local factor at a bad prime p of the L-function in its Euler product
$\alpha_{j,p}$	The Satake parameters of the local factor $L_p(x)$, $j = 1, \ldots d$
$(\mu_j : \nu_k)$	The spectral parameters associated to the L-function
$\gamma(s, (\mu_j : \nu_k))$	The product of the Gamma-factors of the completed L-function
$\Delta(z)$	The Ramanujan Δ-function defined in Chapter 1
$\tau(n)$	The Ramanujan tau function appearing in Chapter 1
$G_n(g(s), \Sigma, N, \{\delta_j; \mu_j\})$	The particular Mellin transform considered in Chapter 2
Σ	In Chapter 2, $\Sigma + \frac{1}{2}$ is the real part of the integration line for the Mellin transform G_n
$\partial\Omega(\Sigma, T)$	The rectangle path chosen in Chapter 2 with vertices at the points $(\frac{1}{2} \pm \Sigma \pm iT)$
$\Omega(\Sigma, T)$	The interior of the rectangle defined in Chapter 2
$\lvert\lvert f(s)\rvert\rvert_{L^p(x+i\mathbb{R})}$	The L^p-norm of a complex function $f(s)$ along $\mathrm{Re}(s) = x$

$\int_{(\sigma)} f(s)ds$	It is $\lim_{t\to\infty} \int_{\sigma-it}^{\sigma+it} f(s)ds = i\int_{-\infty}^{\infty} f(\sigma+it)dt$
$g^e(s), g^o(s)$	The even and odd part of a function $g(s)$ such that $g^e(s)+g^o(s)=g(s)$, see Chapter 2
$g_{k,a,c,\tau}(s)$	The particular choice of test-functions introduced in Chapter 2
$\mu(n)$	The Möbius function defined in Chapter 5
$M(x)$	The Mertens function $M(x)=\sum_{n\le x}\mu(n)$
$Z(t)$	The Hardy Z-function appearing in Chapter 3
Gap_L	The largest gap between zeros of the L-function $L(s)$, see Chapter 3
Gap_*	The largest gap between zeros of any L-function in $\mathcal{S}_{hol}$, defined in Chapter 3
$\Lambda_L(n)$	The generalized von Mangoldt function attached to a generic L-function, see Chapter 3
$\hat{f}(x)$	The Fourier transform of a function f
$B(z)$	The Beurling function appearing in Chapter 3
$\mathrm{sgn}(x)$	The sign function appearing in Chapter 3
$S^{\pm}_{\alpha,\beta;\delta}(z)$	The Selberg minorant and majorant functions introduced in Chapter 3
$G^{\pm}$	The feasible set as in Chapter 3
$c_q(n)$	The Ramanujan sum $c_q(n)=\sum_{(h,q)=1} e^{2\pi inh/q}$
$(h,q^\beta)_\beta=1$	It means that h ranges over the non-positive integers less than q^β such that h and q^β have no common β-th divisor other than 1
$c_q^{(\beta)}(n)$	The generalized Ramanujan sum defined in Chapter 4, $c_q^{(\beta)}(n)=\sum_{(h,q^\beta)_\beta=1} e^{2\pi inh/q^\beta}$
$\sigma_z(n)$	The divisor function $\sigma_z(n)=\sum_{d\mid n} d^{\beta z}$
$\sigma_z^{(\beta)}(n)$	The generalized divisor function $\sigma_z^{(\beta)}(n)=\sum_{d^\beta\mid n} d^{\beta z}$
$\mathfrak{C}^{(\beta)}(n,x)$	The summatory of the generalized Ramanujan sum as in Chapter 4
$\Lambda(n)$	The von Mangoldt function $\Lambda(n)=\sum_{d\delta=n}\mu(d)\log\delta$
$\psi(x)$	The Chebyshev function $\psi(n)=\sum_{n\le x}\Lambda(n)$
$\Lambda_{k,m}^{(\beta)}(n)$	The generalized von Mangoldt function $\Lambda_{k,m}^{(\beta)}(n)=\sum_{a\delta=n} c_d^{(\beta)}(m)\log^k\delta$ defined in Chapter 4
$\psi_m^{(\beta)}(x)$	The generalized von Chebyshev function $\psi_m^{(\beta)}(x)=\sum_{n\le x}\Lambda_{1,m}^{(\beta)}(n)$ defined in Chapter 4
$\varpi_n^{(\beta)}(z)$	The Bartz function defined in Chapter 4

$Q(x)$	In Chapter 5, the number of positive squarefree numbers less or equal to x. In Chapter 6, a polynomial satisfying $Q(0)=1$ and $Q(x)+Q(1-x)=C$, where C is a constant
$M(\mathcal{A},x)$	The Möbius random walk $M(\mathcal{A},x)=\sum_{n\le x}\mu(n)a_n$ for a sequence $\mathcal{A}=(a_n)$
$K(\omega,r_0,\alpha)$	The extended K-class defined in Chapter 5
$\varphi(x),\psi(x)$	In Chapter 5, a pair of reciprocal functions under a certain kernel (cosine, Hankel, . . .)
$Z_1(s),Z_2(s)$	These denote the Mellin transforms of $\varphi(x)$ and $\psi(x)$ respectively, and normalized by a certain factor of $\Gamma(s/2)$
$B_n(x)$	The n-th Bernoulli polynomials
$J_\nu(x)$	The J-Bessel function of the first kind of order ν
$K_\nu(x)$	The K-Bessel function of the second kind of order ν
$\zeta(s,\alpha)$	The Hurwitz zeta-function $\zeta(s,\alpha)=\sum_{n=1}^{\infty}\frac{1}{(n+\alpha)^s}$
${}_1F_1(a,b;x)$	The confluent hypergeometric function
${}_2F_1(a,b,c;x)$	The Gauss hypergeometric function appearing in Chapter 5
${}_pF_q\left(\begin{matrix}a_1 & \dots & a_p\\ b_1 & \dots & b_q\end{matrix};z\right)$	The generalized hypergeometric function
$D_n(x)$	The Weber parabolic cylinder functions as in Chapter 5
$N(T)$	The number of zeros ρ of $\zeta(s)$ for which $0<\beta<1$ and $0\le\gamma\le T$
$N_0(T)$	The number of zeros ρ of $\zeta(s)$ for which $\beta=\frac{1}{2}$ and $0\le\gamma\le T$
$N_0^*(T)$	The number of simple zeros ρ of $\zeta(s)$ for which $\beta=\frac{1}{2}$ and $0\le\gamma\le T$
κ	The proportion of zeros on the critical line, i.e. $\kappa=\liminf_{T\to\infty}N_0(T)/N(T)$
κ^*	The proportion of simple zeros on the critical line, i.e. $\kappa=\liminf_{T\to\infty}N_0^*(T)/N(T)$
$\omega(t)$	The smooth function with compact support appearing in Chapter 6
$d_k(n)$	The number of ways an integer can be written as a product of $k\ge 2$ fixed factors. Also $d_1(n)=1$ and $d_2(n)=d(n)$ denotes the number of divisors of n, see Chapter 6
$\sum_{d\mid n}$	A sum taken over all positive divisors of n
$\sum_{n\le x}f(n)$	A sum taken over all natural numbers not exceeding x; the empty sum being defined as zero
$\sum_{\rho\in\mathcal{B}_\chi}f(\rho)$	A sum taken over nontrivial zeros of an L-function such that the terms are bracketed together according to a bracketing condition $\mathcal{B}_\chi$, see Chapter 5

$\prod_j$	A product taken over all possible values of the index j; the empty product being defined to be unity
$f(x) \sim g(x)$ as $x \to x_0$	This means that $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 1$ with x_0 possibly infinite
$f(x) = O(g(x))$	Landau O-symbol, meaning $\lvert f(x)\rvert \leq C\lvert g(x)\rvert$ for $x \geq x_0$ and some absolute constant $C > 0$
$f(x) \ll g(x)$	This is the same as $f(x) = O(g(x))$
$f(x) \gg g(x)$	This is the same as $g(x) = O(f(x))$
$f(x) \asymp g(x)$	This means that both $f(x) \ll g(x)$ and $f(x) \gg g(x)$ hold
$f(x) = o(g(x))$ as $x \to x_0$	This means that $\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$ with x_0 possibly infinite

Dedicated to my family and my love…

Summary

Despite its unifying content, the theme of the thesis is very modular. The chapters are indeed fairly independent from each other. The central topic behind them are the L-functions, their arithmetic and analytic properties, related computational methods, and their connections to various mathematical objects.

Each chapter contains a detailed introduction and motivation to study its associated topic. We summarize each chapter below.

I. An introduction to the central objects of the thesis, namely the L-functions. The Selberg class is defined, the most important properties are explained and the approximate functional equation is shown.

II. A method to recover Dirichlet coefficients of self-dual L-functions is introduced. Moreover, the python implementation is explained and bounds for the relative error of the computed solution are computed.

III. The upper bound on the largest gap between consecutive zeros of general entire L-functions is improved from 45.3236 to 41.54 under GRH and the Ramanujan hypothesis. Moreover, a new conjecture about the lowest upper bound is stated. This chapter is taken from [KRZ].

IV. Properties of a new arithmetic function generalizing the Ramanujan sum are derived, Moreover, alternative Riemann hypothesis equivalences concerning this new arithmetic function are proved, and additional results about the Bartz function are obtained. This is based on [KR16].

V. A class of functions that satisfies intriguing explicit formulae of Ramanujan and Titchmarsh involving the zeros of an L-function in the Selberg class of degree one and its associated Möbius function is studied. Moreover, some applications with certain explicit examples are obtained. This material is from the first half of [KRR14].

VI. The mollification put forward by Feng is computed by analytic methods and the situation of the percentage of the critical zeros of the Riemann-zeta function on the critical line is explained. This chapter is based on the preprint [KRZ16].

Additional work during this degree but not covered in this thesis can be found in [KR16; KRR14; KRZ16; KRZ].

Chapter 1

Introduction

1.1 The Riemann zeta-function and L-functions

Analytic number theory has been a very active field of research for the last hundred and fifty years. The Riemann zeta-function was introduced by Leonhard Euler, who first noticed its connection to prime numbers by proving in 1737 the *Euler product formula*:

$$\zeta(x) := 1 + \frac{1}{2^x} + \frac{1}{3^x} + \frac{1}{4^x} + \ldots = \sum_{n=1}^{\infty} \frac{1}{n^x} = \prod_{p \text{ prime}} \frac{1}{1-p^{-x}},$$

where x is a real number greater than 1. He used this formula to deduce that primes are infinite (although this was already known and proved by Euclid in the ancient Greece), and moreover that the sum

$$\sum_{p \text{ prime}} \frac{1}{p}$$

diverges. Therefore, he shows that prime numbers are relatively frequent in the set of natural numbers.

It was only with Bernhard Riemann that this function assumed a central role in the analytic number theory. In his 1859 paper [Rie59], instead of taking a real $x > 1$, he took a complex variable $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$ and, using tools of complex analysis that had already been introduced by Cauchy and other mathematicians fifty years earlier, he proved that $\zeta(s)$ could be *analytically continued* to the entire complex plane with a single pole at $s = 1$. This analytic continuation is determined by a certain *functional equation*

$$\Lambda(s) := \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) = \Lambda(1-s),$$

where $\Gamma(s)$ is the Gamma-function

$$\Gamma(s) = \int_0^{+\infty} e^{-x}x^{s-1}dx, \quad \mathrm{Re}(s) > 0,$$

that relates values of the ζ-function at s to values at $1-s$, i.e. values to the left or to the right of the vertical line $\mathrm{Re}(s) = \frac{1}{2}$, which is the *critical line*.

Moreover, Riemann was interested in the complex zeros of the ζ-function. He gave an asymptotic estimate of the number of non-trivial zeros along the critical line and he wrote in his paper that it's "highly likely" that all the non-trivial zeros lie on the critical line $\mathrm{Re}(s) = \frac{1}{2}$. This conjecture is nowadays known as the famous *Riemann hypothesis* and it's so far one of the most important open problems in all of mathematics. Hilbert alreday realized the importance of this conjecture by putting it in his twenty-three *Hilbert's problems* in 1900 as the eighth problem. After one century no one had been able to prove the conjecture, and thus the Clay Mathematics Institute [Bom00], in 2000, put it in the seven Millenium Problems, explaining that a proof of the Riemann hypothesis "would shed light on many of the mysteries surrounding the distribution of prime numbers".

L-functions were first introduced by Dirichlet in 1837 [Dir37], who proved using functions of the form

$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s} \tag{1.1}$$

for s real and $s>1$, that the number of primes in sets of the form $\{a+k\cdot n\ :\ n\in\mathbb{N}\}$ are infinitely many when $(a,k)=1$. The arithmetic function χ is called a *Dirichlet character* modulo k; it arises from completely multiplicative functions on $(\mathbb{Z}/k\mathbb{Z})^*$ and it is extended to all natural numbers by k-periodicity and by defying $\chi(n)=0$ if $(n,k)>1$. Dirichlet proved that the L-function (1.1) did not vanish on the vertical line $\mathrm{Re}(s)=1$ and he used this fact to prove the infiniteness of primes in arithmetic progressions.

Riemann, using the same technique as for the ζ-function, was able to extend $L(s,\chi)$ to complex vaules and prove the existence of the analytic continuation of $L(s,\chi)$. Moreover, he observed that if χ is principal, then the corresponding L-function has a simple pole at $s=1$.

It was soon clear that all properties that characterized the Riemann zeta-function could be proved for the L-function (1.1). For instance, because of the complete multiplicativity of the Dirichlet characters, the Euler product holds

$$L(s,\chi)=\prod_p(1-\chi(p)p^{-s})^{-1}$$

in the half-plane of absolute convergenge $\mathrm{Re}(s)>1$.

Moreover, for primitive characters χ with modulus k, there exists a functional equation of the form

$$\Lambda(s,\chi):=\left(\frac{\pi}{k}\right)^{-(s+a)/2}\Gamma\left(\frac{s+a}{2}\right)L(s,\chi)=\frac{\tau(\chi)}{i^a\sqrt{k}}\Lambda(1-s,\bar{\chi}), \tag{1.2}$$

where $\tau(\chi)$ is the Gauss sum, relating values of $L(s,\chi)$ to values of $L(1-s,\bar{\chi})$.

Dedekind was able to extend Dirichlet's work to number fields. In his 1877 paper [Ded77] he was able to extend the definition of the Dirichlet sum (1.1) to sums over the nonzero ideals of the ring of integers of the number field extension.

It was only with Ramanujan in 1916 [Ram16] that a truly different kind of L-function was discovered. He introduced the Δ-function and expressed it as a Fourier series for $q=e^{2\pi iz}$,

$$\Delta(z)=q\prod_{n=1}^{\infty}(1-q^n)^{24}=\sum_{n=1}^{\infty}\tau(n)q^n. \tag{1.3}$$

He conjectured that $\tau(nm)=\tau(n)\tau(m)$ for m and n coprime, and then deduced a recurrence relation on prime powers

$$\tau(p^{j+1})=\tau(p)\tau(p^j)-p^{11}\tau(p^{j-1}),\ j\in\mathbb{N},\ p\text{ prime},$$

and finally observed that $|\tau(p)|\leq p^{11/2}$ whenever p is prime. The first two statements were proved by Mordell in 1917 [Mor17], and the third one by Deligne in 1974 [Del74] for which he won the Fields Medal.

In 1917, Hecke [Hec17] established the functional equation of the Dedekind zeta-function for any number field. He remarked that for abelian extensions the Dedekind zeta-function can be factored as a product of Dirichlet L-functions.

Hecke, after Mordell's work, was also able to introduce a certain class of operators on the space of cusp forms. The forms which were simultaneously eigenvalues of these operator have multiplicative Fourier coefficients so that their associated Dirichlet series have Euler products and a functional equation.

As for general algebraic varieties over alegraic number fields, Hasse and Weil made the major contributions by defining the Hasse-Weil L-function. In partiular, an L-function coming from an elliptic curve E over $\mathbb{Q}$ and conductor N takes the form

$$L(s,E) = \prod_p L_p(s,E)^{-1},$$

where

$$L_p(s,E) = \begin{cases} 1 - a(p)p^{-s} + p^{1-2s}, & \text{if } p \nmid N, \\ 1 - a(p)p^{-s}, & \text{if } p \mid\mid N, \\ 1, & \text{if } p^2 \mid N, \end{cases}$$

where in the case of good reduction (first case) $a(p) = p + 1 - \#E(\mathbb{F}_p)$ and in the case of multiplicative reduction (second case) $a(p)$ is equal to ± 1 depending upon whether E has split or non-split multiplicative reduction at p. Interestingly, there is another Millenium Problem related to these functions, which is called the *Birch and Swinnerton-Dyer conjecture*. It states that the order of nonvanishing of $L(s,E)$ at $s = 1$ is equal to the rank of E, which is an important invariant of elliptic curves.

The largest known database of L-functions is the LMFDB (L-functions and modular forms database) [LMF16]. It is an online database that groups various examples of L-functions and related objects. The goal of the project is to provide an easily accessible tool that faithfully exhibits the invarant of any L-function and the interconnections between L-functions and other objects.

L-functions could arise from many other objects, but they all share similar Dirichlet series, Euler products over primes and functional equations. In the next section, we will see the axiomatic definition of the class of L-functions.

1.2 The Selberg class

The main properties L-functions should satisfy were axiomatized by Selberg in 1992 [Sel92], where he introduced a general class of L-functions with desirable properties.

1.2.1 Selberg's original definition

Let us set the convention that an empty product is equal to 1. The Selberg class $\mathcal{S}$ consists of functions $F(s)$ of a complex variable s satisfying the following properties:

1. for $\mathrm{Re}(s) = \sigma > 1$, F has Dirichlet series given by

$$F(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}; \tag{1.4}$$

2. for any fixed $\epsilon > 0$, $b(n)$ satisfies the Ramanujan bound
$$b(n) = O(n^{\epsilon}) \tag{1.5}$$
where the implied constant may depend on ϵ;

3. for some integer $m \geq 0$, $(s-1)^m F(s)$ extends to an entire function of finite order;

4. there exist numbers $K > 0$, $Q > 0$, $\alpha_j > 0$, $r_j \in \mathbb{C}$ with $\mathrm{Re}(r_j) \geq 0$ such that
$$\Lambda(s) = Q^s \prod_{j=1}^{K} \Gamma(\alpha_j s + r_j) F(s) = \varepsilon \bar{\Lambda}(1-s), \tag{1.6}$$
where ε is a complex number such that $|\varepsilon| = 1$ and $\bar{\Lambda}(s) = \overline{\Lambda(\bar{s})}$;

5. for σ sufficiently large
$$\log F(s) = \sum_{n=1}^{\infty} \frac{a(n)}{n^s}, \tag{1.7}$$
where $a(n) = 0$ unless n is positive power of a prime, and $a(n) = O(n^{\theta})$ for some $\theta < 1/2$.

The last axiom (1.7) implies that each L-function can be written as a product over primes
$$F(s) = \prod_p F_p(s),$$
where
$$\log F_p(s) = \sum_{k=0}^{\infty} \frac{a(p^k)}{p^{ks}}.$$
In particular, from this it follows that the Dirichlet coefficients $b(n)$ are multiplicative, and that we can always write an L-function as
$$F(s) = \prod_p \sum_{k=0}^{\infty} b(p^k) p^{-ks}.$$

There is an important invariant associated to an L-function. For a non-negative integer n, the H*-invariants* are defined by
$$H_F(n) = 2 \sum_{j=1}^{K} \frac{B_n(r_j)}{\alpha_j^{n-1}},$$
where $B_n(x)$ are the familiar n-th order Bernoulli polynomials. The first few $B_n(x)$'s are given by
$$B_0(x) = 1, \quad B_1(x) = x - \frac{1}{2}, \quad B_2(x) = x^2 - x + \frac{1}{6}, \quad \cdots.$$
Hence we find that
$$H_F(0) = 2 \sum_{j=1}^{K} \alpha_j =: d_F, \quad H_F(1) = 2 \sum_{j=1}^{K} (r_j - 1/2), \quad \cdots, \tag{1.8}$$

where d_F is the *degree* of the L-function.

The conductor q_F of an L-function is defined by

$$q_F = (2\pi)^{d_F} Q^2 \prod_{j=1}^{K} \alpha_j^{\alpha_j}. \tag{1.9}$$

The degree d_F and conductor q_F are conjectured to be positive integers.

Conjecture 1.2.1 (Degree and conductor conjecture)**.** *$d_F \geq 0$ and $q_F \geq 1$ are positive integers.*

The case $d_F = 0$ occurs precisely when $F \equiv 1$ was proved by Conrey and Ghosh [CG93], while Kaczorowski and Perelli [KP99; KP02] proved the degree conjecture for $0 < d_F < 1$ and $1 < d_F < 5/3$, and they later improved their result to $d_F \leq 2$ [KP11].

Moreover, it is expected that the spectral parameters α_j in (1.6) are all integers or half-integers, as we will see later in this chapter.

Assuming Conjecture 1.2.1, we can thus partition $\mathcal{S}$ into

$$\mathcal{S} = \bigcup_{d \geq 0} \mathcal{S}_d$$

where $\mathcal{S}_d$ is the class containing the L-functions of degree d.

The following result is Theorem 3 from Kaczorowski and Perelli [KP99] using the H-invariant definition (1.8).

Lemma 1.2.1. *Let $F \in \mathcal{S}$. Suppose that $d_F = 1$ and $\operatorname{Re}(H_F(1))$ is either 0 or 1. If $q_F = 1$ then $F(s) = \zeta(s)$. If $q_F \geq 2$ then there exists a primitive Dirichlet character $\chi \bmod q_F$ with $\chi(-1) = -(2\operatorname{Re}(H_F(1)) + 1)$ such that $F(s) = L(s + i\operatorname{Im}(H_F(1)), \chi)$.*

$\mathcal{S}_1$ is therefore fully characterized.

Corollary 1.2.1. *$\mathcal{S}_1$ is the class containing the Riemann zeta-function and all shifts of Dirichlet L-functions $L(s + i\theta, \chi)$ (1.1), where $\theta \in \mathbb{R}$ attached to a primitive, nonprincipal Dirichlet character χ modulo q.*

Unfortunatley, the class becomes exponentially complicated to characterize already from $d = 2$. For example, there are various sources generating degree 2 L-functions such as holomorphic cusp forms, elliptic curves, Maass forms for $GL(2)$, Dedekind zeta functions for quadratic number fields and Artin representations of dimension 2. All these sources provide a vast set of L-functions of degree 2, yet nobody knows if it is a complete one.

The holomorphicity of the L-functions except possibly at $s = 0$ or $s = 1$ implies that there are zeros of $L(s)$ which arise from the poles of the Gamma-terms in (1.6). We call these zeros the *trivial zeros*. The assumption $\operatorname{Re}(r_i) \geq 0$ in (1.6) for the parameters in equation (1.6) guarantees that these zeros lie outside the region $0 < \sigma < 1$, which is called the *critical strip*. The zeros lying in this vertical strip are the *non-trivial zeros*.

The functional equation (1.6) is centered on the vertical line $\operatorname{Re}(s) = \frac{1}{2}$. As for the Riemann Hypothesis and the Generalized Riemann Hypothesis for Dirichlet L-functions, we have the following generalization to every element in the Selberg class.

Conjecture 1.2.2 (Grand Riemann Hypothesis)**.** *Suppose that $\rho = \beta + i\gamma$ is a zero of $F \in \mathcal{S}$ with $0 < \beta < 1$, then $\beta = \frac{1}{2}$.*

Despite believed to be true by many mathematicians, the Grand Riemann Hypothesis has not been proved yet. Even a hypothetical proof for a single element of the Selberg class would be an achievement of historical measure.

1.2.2 Farmer's definition

Selberg's original definition is however too general, since all known examples of L-functions in the online database LMFDB [LMF16] have more restrictive properties, in particular about the spectral parameter range. Farmer [Far12; FKL12; FKL15] and other mathematicians were able to provide a much more faithful definition of the Selberg class assuming the degree and conductor conjecture (Conjecture 1.2.1) and the relations between the parameters involved. According to Farmer's definition, an L-function $L(s)$ has to satisfy these properties:

1. *(Dirichlet series)* it can be expressed as a formal Dirichlet series
$$L(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}, \tag{1.10}$$
where $b(1) = 1$ which is absolutely convergent for $\mathrm{Re}(s) > 1$ and thus it defines an analytic function in this domain;

2. *(Ramanujan hypothesis)* their Dirichlet coefficients $b(n)$ satisfy the following Ramanujan hypothesis, for any $\epsilon > 0$,
$$b(n) = O\left(n^{\epsilon}\right); \tag{1.11}$$
where the implicit constant may depend on ϵ.

3. *(Analytic continuation)* there exists an analytic continuation that extends $L(s)$ to the whole complex plane with a possible pole at $s = 1$ of order m. In other words, $(s-1)^m L(s)$ is an entire function of finite order;

4. *(Functional equation)* there exists a functional equation, relating $L(s)$ to $\bar{L}(1-s)$ of the following form
$$\begin{aligned}\Lambda(s) &= N^{s/2} \prod_{j=1}^{d_1} \Gamma_{\mathbb{R}}(s+\mu_j) \prod_{k=1}^{d_2} \Gamma_{\mathbb{C}}(s+\nu_k) L(s) = N^{s/2} \gamma(s, (\mu_j \; : \nu_k)) L(s) \\ &= \varepsilon \bar{\Lambda}(1-s),\end{aligned} \tag{1.12}$$
where
$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right), \quad \Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s} \Gamma(s). \tag{1.13}$$
The invariant $d = d_1 + 2d_2$ is the *degree* of the L-function and $N \in \mathbb{N}$ is the *level* or *conductor*. μ_j, ν_k are called *spectral parameters* and they satisfy $\mathrm{Re}(\mu_j) \in \{0, 1\}$, $\mathrm{Re}(\nu_k)$ are integers or half-integers . Moreover, $2\sum_j \mu_j + \sum_k \nu_k$ is a positive real number. Finally, $\varepsilon \in \mathbb{C}$ is the *root number* or *sign* and satisfies $|\varepsilon| = 1$;

5. *(Euler product)* the L-function can be expressed as an Euler product over prime numbers,
$$L(s) = \prod_{p|N} \tilde{L}_p(p^{-s})^{-1} \prod_{p\nmid N} L_p(p^{-s})^{-1}, \tag{1.14}$$

where the local factors at good primes L_p can be expressed as $L_p(x) = 1 - b(p)x + \ldots + (-1)^d \chi(p)x^d$, and at bad primes $\tilde{L}_p(x)$ is a polynomial of degree less than d with $\tilde{L}_p(0) = 1$. Moreover, the polynomials $L_p(s)$ can be factored into

$$L_p(x) = \prod_{j=1}^{d} (1 - \alpha_{j,p} x), \tag{1.15}$$

and the numbers $\alpha_{j,p}$ are called Satake parameters. Because of (1.11), they satisfy $|\alpha_{j,p}| = 1$.

We call the 4-tuple of parameters $(d, N, (\mu_j : \nu_k)), \varepsilon)$ the *Selberg data* of an L-function.

The condition $2\sum_j \mu_j + \sum_k \nu_k \in \mathbb{R}_+$ ensures that the L-function is centered on the real axis, and thus it avoids treating unwanted vertical shifts $L_\theta(s) = L(s + i\theta)$ that are uninteresting. Moreover, the positivity assumption makes sure that trivial zeros don't lie in the critical strip, therefore contradicting Conjecture 1.2.2 trivially.

The following table summarizes the differences between the two definitions:

Axioms/properties	Selberg's definition	Farmer's definition
Dirichlet series	$F(s) = \sum_{n=1}^{\infty} b(n) n^{-s}$	$L(s) = \sum_{n=1}^{\infty} b(n) n^{-s}$
Analytic continuation	$\exists m \geq 0$: $(s-1)^m F(s)$ entire	$\exists m \geq 0$: $(s-1)^m L(s)$ entire
Completed L-function	$\Lambda(s) = Q^s \prod_{j=1}^{K} \Gamma(\alpha_j s + r_j) F(s)$ $Q > 0$, $\alpha_j \geq 0$, $r_j \in \mathbb{C}$	$\Lambda(s) = N^{s/2} \prod_{j=1}^{d_1} \Gamma_{\mathbb{R}}(s + \mu_j)$ $\times \prod_{k=1}^{d_2} \Gamma_{\mathbb{C}}(s + \nu_k) L(s)$ $N \in \mathbb{N}$, $\mathrm{Re}(\mu_j) \in \{0, 1\}$, $\mathrm{Re}(\nu_k) \in \mathbb{N}/2$, $2\sum_j \mu_j + \sum_k \nu_k \in \mathbb{R}_+$
Functional equation	$\Lambda(s) = \varepsilon \bar{\Lambda}(1-s)$ $\lvert\varepsilon\rvert = 1$	$\Lambda(s) = \varepsilon \bar{\Lambda}(1-s)$ $\lvert\varepsilon\rvert = 1$
Ramanujan hypothesis	$b(n) = O(n^{\epsilon})$ for all $\epsilon > 0$	$b(n) = O(n^{\epsilon})$ for all $\epsilon > 0$
Euler product	$F(s) = \prod_p F_p(s)$ $\log F_p(s) = \sum_{k=0}^{\infty} \frac{a(p^k)}{p^{ks}}$ $a(n) \ll n^{\theta}$, $\theta < 1/2$	$\prod_{p \mid N} \tilde{L}_p(p^{-s})^{-1} \prod_{p \nmid N} L_p(p^{-s})^{-1}$ $L_p(x)$ of degree $d = d_1 + 2d_2$ $\tilde{L}_p(x)$ of degree $\leq d - 1$
Vertical shifts	Yes	No
Closed under multiplication	Yes	Yes
Conductor and degree conjecture	Not assumed	Assumed

Although Farmer's definition is more detailed, especially in the functional equation axiom (1.12), for certain purposes it is still better to use the original Selberg's original definition of the Selberg class. We will use for example special cases of Farmer's definition in Chapter 2 and Chapter 3, while for certain invariants defined in Chapter 5 we will use the general definition provided by Selberg.

1.2.3 Self-dual and conjugate pairs L-functions

An L-function is called *self-dual* if its Dirichlet coefficients $b(n)$ are real. In particular, it implies that the corresponding sign is real: $\varepsilon = \pm 1$. As a trivial consequence, its Satake parameters are either $\alpha_{p,j} = \pm 1$, or they appear in conjugate pairs. There is always a nontrivial central zero $L(1/2) = 0$ if the L-function is self-dual with $\varepsilon = -1$.

Another subclass of particular importance comes from L-functions with spectral parameters in *conjugate pairs*, i.e. such that they are invariant under conjugation: $\overline{\{\mu_j\}} = \{\mu_j\}$ and $\overline{\{\nu_k\}} = \{\nu_k\}$. These L-functions satisfy

$$\bar{\gamma}(s, (\mu_j \ : \nu_k)) = \gamma(s, (\mu_j \ : \nu_k)),$$

and every self-dual L-function has spectral parameters that come in conjugate pairs, yet not every L-function which has spectral parameters in conjugate pairs is self-dual.

1.2.4 The motivic weight

Occasionally, L-functions arise from objects where their natural contruction doesn't belong to the Selberg class itself; instead they need to be suitably normalized.

For example, if we consider the Ramanujan Δ-function (1.3), a cusp newform of weight $k = 12$ for the full modular group $SL(2, \mathbb{Z})$, its natural Dirichlet series coming from the Fourier coefficients $\tau(n)$ of $\Delta(z)$ at infinity is given by

$$L(s, \Delta) := \sum_{n=1}^{\infty} \frac{\tau(n)}{n^s} \tag{1.16}$$

for $\mathrm{Re}(s) > k - 1$, where $\tau(1) = 1$; its functional equation takes the simple form

$$\Lambda(s, \Delta) = \Gamma_{\mathbb{C}}(s) L(s, \Delta) = \Lambda(k - s, \Delta).$$

which means that the critical line lies at $\mathrm{Re}(s) = \frac{k-1}{2}$, and thus fails to belong in the Selberg class, in this form.

To avoid this problem, one defines the *motivic weight* of an L-function *arithmetically normalized*

$$L_{arithmetic}(s, f) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s},$$

which comes naturally from the object f to be $\omega \in \mathbb{N}$ such that the *analytically normalized* Dirichlet series

$$L_{analytic}(s, f) = L_{arithmetic}(s + \omega/2, f) = \sum_{n=1}^{\infty} \frac{b(n) n^{-\omega/2}}{n^s}$$

has critical line $\mathrm{Re}(s) = \frac{1}{2}$ and functional equation of the form

$$\Lambda_{analytic}(s, f) = \varepsilon \bar{\Lambda}_{analytic}(1 - s, f).$$

Common examples of motivic weight 0 L-functions are the Riemann zeta-function $\zeta(s)$, Dirichlet L-functions associated to a primitive Dirichlet character $L(s,\chi)$, the Dedekind zeta function $\zeta_K(s)$ and L-functions coming from $GL(d)$ Maass forms of weight 0 (see §1.5 and §1.6).

L-functions $L(s,E)$ associated to elliptic curves over $\mathbb{Q}$ are of motivic weight $\omega = 1$, while for L-functions $L(s,f)$ coming from cusp forms of weight k, the motivic weight is $\omega = k-1$.

We say that an L-function in its analytic normalization is *arithmetic* with respect to the motivic weight ω if their normalized Dirichlet coefficients $b(n)n^{\omega/2}$ are algebraic integers. Most L-functions are known to be arithmetic, except (conjecturally) for those ones coming from $GL(d)$ Maass forms associated to nontrivial eigenvalues, see §1.5.

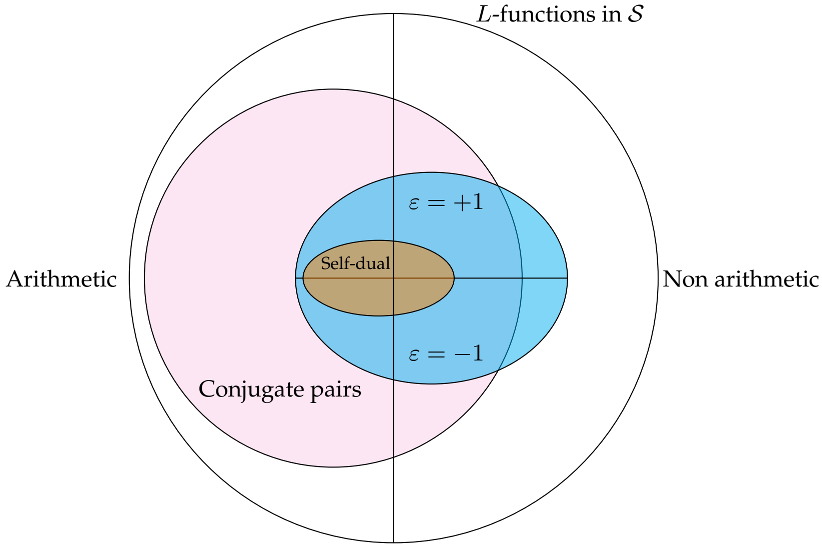

FIGURE 1.1: The structure of the Selberg class $\mathcal{S}$ and its subclasses. The "Conjugate pairs" set denotes all L-functions with spectral parameters that come in conjugate pairs.

1.2.5 Primitive L-functions

There exists a subclass of L-functions of particular interest. If $L_1(s)$ and $L_2(s)$ are two L-functions according to Farmer's definition, with Selberg data $(d_1, N_1, (\mu_{1,j_1} : \nu_{1,k_1}), \varepsilon_1)$ and $(d_2, N_2, (\mu_{2,j_2} : \nu_{2,k_2}), \varepsilon_2)$ respectively, then their product $L(s) = L_1(s)L_2(s)$ is an L-function with Selberg data

$$(d_1 + d_2, N_1N_2, (\mu_{1,j_1}, \mu_{2,j_2} : \nu_{1,k_1}, \nu_{2,k_2}), \varepsilon_1\varepsilon_2). \tag{1.17}$$

Thus, L-functions may arise as products of simpler ones. We say that an L-function $L(s)$ is *primitive* if whenever $L(s) = L_1(s)L_2(s)$ with $L_1, L_2 \in \mathcal{S}$, then either $L_1(s) = 1$ or $L_2(s) = 1$.

Therefore, any L-function $L \in \mathcal{S}$ can be factored into a product $L(s) = \prod_j L_j(s)^{e_j}$, where each $L_j(s)$ is a primitive L-function. Uniqueness, instead, is still a conjecture.

Conjecture 1.2.3 (Unique Factorization)**.** *Factorization into primitive L-functions is unique.*

Uniqueness is guaranteed if the Dirichlet coefficients of L-functions at all but finitely many primes coincide.

Theorem 1.2.1 (Strong multiplicity one, p. 393 of [Gol06]). *Let $L_1(s)$ and $L_2(s)$ be two L-functions with Dirichlet coefficients $b_1(n)$ and $b_2(n)$. If*

$$b_1(p) = b_2(p)$$

for all but finitely many p, then $L_1 \equiv L_2$.

Selberg conjectured that primitive L-functions are quasi-orthonormal with respect to a certain inner product.

Conjecture 1.2.4 (Selberg orthonormality conjecture, SOC). *Let $L_1(s)$ and $L_2(s)$ be primitive L-functions. Then*

$$\sum_{p \leq x} \frac{b_{L_1}(p)\overline{b_{L_2}(p)}}{p} = (\delta_{L_1,L_2} + o(1)) \log\log x \tag{1.18}$$

as $x \to +\infty$, where $\delta_{L_1,L_2} = 1$ if $L_1 \equiv L_2$ and $\delta_{L1,L_2} = 0$ otherwise.

Despite the clear statement, checking primitivity using the above equation is hard as the corresponding sum is generally very slowly convergent. It is in fact more reliable to check that all feasible decompositions of the Selberg data into two parts as in (1.17) cannot come from L-functions of lower degrees.

There are however important consequences of the Selberg orthonormality conjecture.

Theorem 1.2.2 (Murty, Conrey, Ghosh). *Assume SOC. Then*

1. *The Unique Factorization conjecture holds,*
2. *$\zeta(s)$ is the only primitive L-function with a pole at $s = 1$,*
3. *$L(1 + it) \neq 0$ for each $t \in \mathbb{R}$ and any element $L \in \mathcal{S}$.*

Sarnak conjectured that primitive L-functions are countable.

Conjecture 1.2.5 (Sarnak countability conjecture). *Let $\mathcal{P}$ be the subclass of primitive L-functions in $\mathcal{S}$. Then $\mathcal{P}$ is countable.*

1.3 The approximate functional equation

The approximate functional equation is a formula which is widely used to compute values of L-functions inside the critical strip where the Dirichlet series of $L(s)$ fails to converge. There are several versions of the equation and the two most common ones are described in this section.

These formulae can be used to find primitive and generic L-functions as well. Methods have been investigated by Dokchitser [Dok04], Rubinstein [Rub05], Molin [Mol10], and further by Farmer, Koutsoliotas and Lemurell [Far12; FKL12; FKL15] for L-functions of degree $d \leq 4$. In particular, Farmer, Koutsoliotas and Lemurell [FKL12] were able to find a certain number of primitive L-functions coming from $GL(3)$ and $GL(4)$ Maass forms using this formulas in a brilliant way.

The following formulation of the approximate functional equation for an L-function comes from Iwaniec and Kowalski [IK04, p. 98]. Here is a slighly modified version of the theorem.

Theorem 1.3.1 (Iwaniec-Kowalski, [IK04]). *Let $L(s)$ be an L-function having properties (1.10)-(1.14). Let $g(z)$ a function which is holomorphic and bounded in the strip $-4 < \mathrm{Re}(z) < 4$, even, and normalized by $g(0) = 1$. Let $X > 0$, then for s in the strip $0 \leq \sigma \leq 1$ we have*

$$\begin{aligned} L(s) =& \frac{1}{\gamma(s,(\mu_j \,:\, \nu_k))}\sum_{n=1}^{\infty}\frac{b(n)}{n^s}G_s\left(\frac{n}{X\sqrt{N}}\right) + \frac{\varepsilon N^{\frac{1}{2}-s}}{\gamma(s,(\mu_j \,:\, \nu_k))}\sum_{n=1}^{\infty}\frac{\bar{b}(n)}{n^s}\bar{G}_s\left(\frac{nX}{\sqrt{N}}\right) \\ &+ \left(\mathrm{Res}_{z=1-s} + \mathrm{Res}_{z=-s}\right)\Lambda(s+z)\frac{g(z)}{z}X^z, \end{aligned} \tag{1.19}$$

where

$$G_s(y) = \frac{1}{2\pi i}\int_{(3)} y^{-z}g(z)\gamma(s+z,(\mu_j \,:\, \nu_k))\frac{dz}{z}.$$

In particular, the last term vanishes if $\Lambda(s)$ is entire.

The rate of decay of the Mellin transform $G_s(y)$ guarantees absolute convergence of both sums in the critical strip $0 \leq \sigma \leq 1$.

Proposition 1.3.1 (Iwaniec-Kowalski, [IK04]). *Suppose $\max\{\mathrm{Re}(s+\mu_j), \mathrm{Re}(s+\nu_k+1)\} \geq 3\alpha > 0$ for all spectral parameters in (1.12). Then the derivatives of $G_s(y)$ satisfy*

$$y^a G_s^{(a)}(y) \ll (1+y)^{-A},$$

$$y^a G_s^{(a)}(y) = \delta_a + O\left(y^{-\alpha}\right),$$

where $A > 0$, $\delta_0 = 1$, $\delta_a = 0$ if $a > 0$, and the implied constants depend only on α, a, A and d.

Rubinstein [Rub05] was able to provide a modified version of the approximate functional equation, more suitable for computational purposes. This theorem was extensively used by Farmer [Far12; FKL12; FKL15] and others.

Theorem 1.3.2 (Rubinstein, §3.2 of [Rub05]). *Let $g : \mathbb{C} \to \mathbb{C}$ be an entire function that, for fixed s, satisfies*

$$\left|\Lambda(z+s)g(z+s)z^{-1}\right| \to 0$$

as $|\mathrm{Im}(z)| \to \infty$ in vertical, bounded strips $-\alpha \leq \mathrm{Re}(z) \leq \alpha$. For $s \notin \{0,1\}$ and $L(s) \in \mathcal{S}$,

$$\begin{aligned} \Lambda(s)g(s) =& \frac{r_0 g(0)}{s} + \frac{r_1 g(1)}{s-1} + N^{\frac{s}{2}}\sum_{n=1}^{\infty}\frac{b(n)}{n^s}f_1(s,n) \\ &+ \varepsilon N^{\frac{1-s}{2}}\sum_{n=1}^{\infty}\frac{\overline{b(n)}}{n^{1-s}}f_2(1-s,n), \end{aligned} \tag{1.20}$$

where r_0, r_1 are the residues of $\Lambda(s)$ at 0 and 1, respectively, and

$$f_1(s,n) = \frac{1}{2\pi i}\int_{(\delta)} \gamma(s+z,(\mu_j : \nu_k))g(s+z)z^{-1}\left(\frac{\sqrt{N}}{n}\right)^z dz,$$

$$f_2(1-s,n) = \frac{1}{2\pi i}\int_{(\delta)} \bar{\gamma}(z+1-s,(\mu_j : \nu_k))g(s-z)z^{-1}\left(\frac{\sqrt{N}}{n}\right)^z dz,$$

for $\delta > \max\{0, -\mathrm{Re}(\mu_1/2+s), \ldots, -\mathrm{Re}(\mu_{d_1}/2+s), -\mathrm{Re}(\nu_1+s), \ldots, -\mathrm{Re}(\nu_{d_2}+s)\}$.

1.4 Unconditional bounds on the coefficients

The Ramanujan hypothesis ((1.5) or (1.11)) is an axiom of the Selberg class, although it was only proved for limited subclasses of L-functions. Degree 1 L-functions satisfy the bound trivially, since $|\chi(n)| \leq 1$ for any $n \in \mathbb{N}$ and for any primitive Dirichlet character modulo q.

As for degree 2, Deligne [Del74] proved the Ramanujan hypothesis for holomorphic cusp forms, and more in general for any automorphic cusp form of $GL(n, F)$, where F is a complex multiplication field.

For many other objects which are sources of L-functions, it still remains a conjecture. In particular, for L-functions coming from $GL(d)$ Maass forms the bound is still unproved. Kim and Sarnak ([Kim03], 2003) established the current world record for $GL(2)$ Maass forms. The following theorem is a particular case of the statement originally proved by Kim and Sarnak.

Theorem 1.4.1 (Kim-Sarnak, [Kim03])**.** *Let $L(s, f)$ be an L-function arising from a Maass form for $SL(2, \mathbb{Z})$ with Euler product of the form*

$$L(s) = \prod_p (1 - \alpha_{p,1} p^{-s})^{-1} (1 - \alpha_{p,2} p^{-s})^{-1}.$$

Then its corresponding Satake parameters satisfy the unconditional bounds

$$|\alpha_{p,1}| \leq p^{\frac{7}{64}}, \quad |\alpha_{p,2}| \leq p^{\frac{7}{64}}.$$

Moreover,

$$|b(p)| \leq 2p^{\frac{7}{64}}$$

for all primes p.

The best unconditional bound for the Dirichlet coefficients for L-functions is due to Luo, Rudnick and Sarnak [LRS99]. Weaker bounds were previously obtained by Jacquet and Shalika (1981), who proved that $|\alpha_{p,j}| \leq p^{1/2}$ for L-functions arising from $GL(d)$ Maass forms. The following theorem is a particular case of the result originally proved by Luo, Rudnick and Sarnak.

Theorem 1.4.2 (Luo-Rudnick-Sarnak, [LRS99])**.** *Fix $d \geq 2$. Let $L(s, f)$ be an L-function of degree d coming from a $SL(d, \mathbb{Z})$ Maass form. Then its Satake parameters $\alpha_{p,j}$ satisfy*

$$|\alpha_{p,j}| \leq p^{\frac{1}{2} - \frac{1}{d^2+1}},$$

for all primes p, $1 \leq j \leq d$. Moreover it implies that

$$|b(p)| \leq dp^{\frac{1}{2} - \frac{1}{d^2+1}} < dp^{\frac{1}{2}},$$

for all primes p and $d \geq 1$.

1.5 Classical Maass forms and their L-functions

Maass forms are nonholomorphic functions that satisfy the modularity condition (A.54), moderate growth at infinity and that are eigenfunctions of the hyperbolic Laplacian.

Maass' original motivation was to generalize Hecke's result to real quadratic fields. Hecke showed that he could construct holomorphic modular forms associated to characters of $\mathbb{Q}(\sqrt{d})$ for $d < 0$. The case $d > 0$ was left open until Maass realized that there cannot be a holomorphic function that could work in the same way.

In particular, he observed that the functional equation of the completed L-function of a quadratic field has one $\Gamma_{\mathbb{C}}$-factor if the field is imaginary quadratic, and two $\Gamma_{\mathbb{R}}$-factors if it is real quadratic. If the modular function associated to a real quadratic field were holomorphic, then its functional equation would have had just one $\Gamma_{\mathbb{C}}$-factor. He then observed that the Mellin transform of the K-Bessel function was a product of two $\Gamma_{\mathbb{R}}$-factors, and used this result to defined a new class of modular functions.

L-functions arising from Maass forms are conjecturally the only known ones that are non-arithmetic. This fact motivates the numerical approach through computational methods for finding the corresponding Dirichlet coefficients and the invariants associated to them, as we will see in Chapter 2.

Let $L^2(SL(2,\mathbb{Z})/\mathbb{H})$ be the completion of the space consisting of all smooth functions $f : SL(2,\mathbb{Z})/\mathbb{H} \to \mathbb{C}$ satisfying the L^2-integrability condition

$$\iint_{SL(2,\mathbb{Z})/\mathbb{H}} |f(z)|^2 \frac{dxdy}{y^2} < \infty.$$

This is actually a Hilbert space with respect to the *Petersson inner product*

$$< f, g >= \iint_{SL(2,\mathbb{Z})/\mathbb{H}} f(z)\overline{g(z)} \frac{dxdy}{y^2} \tag{1.21}$$

for all $f, g \in L^2(SL(2,\mathbb{Z})/\mathbb{H})$.

A *Maass form* of type $\nu \in \mathbb{C}$ for $SL(2,\mathbb{Z})$ is a non-zero function $f \in L^2(SL_2(\mathbb{Z})/\mathbb{H})$ which satisfies

1. $f(\gamma \cdot z) = f(z)$ for every $\gamma \in SL(2,\mathbb{Z})$,
2. $\Delta_{\mathbb{H}} f = \nu(1-\nu) f$,
3. $\int_0^1 f(z)dx = 0$,

where $\Delta_{\mathbb{H}}$ is the hyperbolic Laplacian defined as

$$\Delta_{\mathbb{H}} = y^2 \left(\frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y} \right). \tag{1.22}$$

In particular, $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ so that $f(z+1) = f(z)$ and f can be written in the following form, for $z = x + iy$,

$$f(z) = \sum_{m \neq 0} A_m(y) e^{2\pi i x},$$

where

$$A_m(y) = b(m)\sqrt{2\pi y} K_{\nu - \frac{1}{2}}(2\pi |m| y)$$

for certain complex coefficients $b(m) \in \mathbb{C}$.

We introduce T_{-1}, a map from the space of Maass forms of type ν into itself. It is defined through

$$T_{-1} f(x + iy) = f(-x + iy),$$

and invariant under the hyperbolic Laplacian. Then f is said to be *even* if $T_{-1}f = f$ and *odd* if $T_{-1}f = -f$. Furthermore, we can prove that $a(n) = a(-n)$ if f is an even Maass form, and that $a(n) = -a(-n)$ if f is an odd Maass form.

In a similar way, we can define for every integer $n \geq 1$ the so-called *Hecke operators* on $L^2(SL_2(\mathbb{Z})/\mathbb{H})$ by

$$T_n f(z) = \frac{1}{\sqrt{n}} \sum_{\substack{ad=n \\ 0\leq b<d}} f\left(\frac{az+b}{d}\right).$$

These are well-defined and self-adjoint with respect to the Petersson inner product: $< T_n f, g >=< f, T_n g >$. Of particular interest are normalized Maass forms which are eigenfunction of the Hecke operators for every $n \geq 1$, because their Dirichlet coefficients are then multiplicative with $b(1) = 1$, and $b(1) = 0$ if and only if the Maass form is the zero function. In fact,

$$T_n f = b(n) f$$

and the multiplicativity relation follows from the properties of the Hecke operators.

Let f be a normalized Maass form which is an eigenvalue of all Hecke operators, and assume that it is either even or odd. Then we can define

$$L(s,f) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}.$$

It can be shown that $L(s, f)$ has an Euler product of the form

$$L(s,f) = \prod_p (1 - b(p)p^{-s} + p^{-2s})^{-1} \tag{1.23}$$

and it has the following functional equation with two Gamma-factors

$$\Lambda(s,f) = \pi^s \Gamma\left(\frac{s+\epsilon-\frac{1}{2}+\nu}{2}\right)\Gamma\left(\frac{s+\epsilon+\frac{1}{2}-\nu}{2}\right)L(s,f) = (-1)^{\omega}\Lambda(1-s,f), \tag{1.24}$$

where $\omega = 0$ if f is even and $\omega = 1$ if f is odd. Define $\mu \in \mathbb{R}$ to be $i\mu = \nu - \frac{1}{2}$, then using the $\Gamma_{\mathbb{R}}$ notation we can rewrite the above equation as

$$\tilde{\Lambda}(s,f) = \Gamma_{\mathbb{R}}(s+\epsilon+i\mu)\,\Gamma_{\mathbb{R}}(s+\epsilon-i\mu)\,L(s,f) = (-1)^{\epsilon}\tilde{\Lambda}(1-s,f), \tag{1.25}$$

and $\tilde{\Lambda}$ can be identified with Λ. Moreover, $\Lambda(s, f)$ is entire and bounded on vertical strips, and thus the corresponding L-function belongs to the Selberg's class according to Farmer's definition.

In the same way, we can define Maass forms of level N associated to the congruence subgroup $\Gamma_0(N)$ and obtain a functional equation of the form

$$\Lambda(s,f) = N^{s/2}\Gamma_{\mathbb{R}}(s+\epsilon+i\mu)\,\Gamma_{\mathbb{R}}(s+\epsilon-i\mu)\,L(s,f) = \omega_{\epsilon,N}\Lambda(1-s,f), \tag{1.26}$$

and a similar Euler product, where the Euler factors $\tilde{L}_p(p^{-s})$ at primes p dividing N are either equal to $1 - b(p)p^{-s}$ or equal to 1.

We denote the eigenvalue of a Maass form f by $\lambda = \nu(1-\nu)$. Selberg [Sel65] conjectured that the smallest eigenvalue of the hyperbolic Laplacian on the modular group is real and greater or equal than $\frac{1}{4}$.

Conjecture 1.5.1 (Selberg Eigenvalue Conjecture, [Sel65]). *The smallest eigenvalue λ_1 of $\Delta_{\mathbb{H}}$ on the modular group $\Gamma_0(N) \setminus \mathbb{H}$ satisfies*

$$\lambda_1(\Gamma_0(N) \setminus \mathbb{H}) \geq \frac{1}{4}$$

for any $N \geq 1$.

We can write the eigenvalue in the form $\lambda = \nu(1-\nu) = 1/4+\mu^2$ using the notation as before, where one usually refers to μ as the "eigenvalue" of f. The smallest eigenvalue of a Maass form of full level $N = 1$ is known to be $\mu = 9.53369526135$, corresponding to an odd Maass form (see [LMF16, $GL(2)$ Maass forms]).

1.6 Maass Forms for $GL(3)$ and $GL(d)$

The following section is mostly taken from [Gol06, Chapter 6].

In order to define $GL(3)$ and $GL(d)$ with $d > 3$ Maass forms, we need to generalize the upper-half plane to higher dimensions. Unfortunately, unlike the 2 dimensional case, the generalized upper-half plane doesn't have a complex structure and it is defined in a different way than $\mathbb{H}$.

The three-dimensional upper-half plane $\mathbb{H}_3$ is defined as the set of all matrices $\mathbf{z} = \mathbf{x} \cdot \mathbf{y}$, where

$$\mathbf{x} = \begin{pmatrix} 1 & x_{1,2} & x_{1,3} \\ 0 & 1 & x_{2,3} \\ 0 & 0 & 1 \end{pmatrix} \qquad \mathbf{y} = \begin{pmatrix} y_1 y_2 & 0 & 0 \\ 0 & y_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with $x_{1,2}, x_{1,3}, x_{2,3} \in \mathbb{R}$ and $y_1, y_2 > 0$. Explicitly, every $\mathbf{z} \in \mathbb{H}_3$ can be written in the form

$$\mathbf{z} = \begin{pmatrix} y_1 y_2 & x_{1,2} y_1 & x_{1,3} \\ 0 & y_1 & x_{2,3} \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.27}$$

The above definition is due to the Iwasawa decomposition as the quotient space

$$\mathbb{H}_3 \cong GL(3)/(O(3) \cdot \mathbb{R}^\times), \tag{1.28}$$

meaning that each $\mathbf{g} \in GL(3)$ can be written in the form

$$\mathbf{g} = \mathbf{z} \cdot \mathbf{k} \cdot \mathbf{d},$$

where $\mathbf{z} \in \mathbb{H}_3$ is uniquely determined, $\mathbf{k} \in O(3)$ and $\mathbf{d} \in Z_3$ is a non-zero diagonal matrix which lies in the center of $GL(3)$, thus $Z_3 \cong \mathbb{R}^\times$.

We can prove that the left-invariant $GL(3)$-measure $d^*\mathbf{z}$ on $\mathbb{H}_3$ can be given via the following formula, for $\mathbf{z}$ in the form (1.27),

$$d^*\mathbf{z} = dx_{1,2}dx_{1,3}dx_{2,3}\frac{dy_1 dy_2}{(y_1 y_2)^3},$$

and $f : \mathbb{H}_3 \to \mathbb{C}$ is said to be in $L^2(SL(3,\mathbb{Z})/\mathbb{H}_3)$ if and only if $\int_{SL(3,\mathbb{Z})/\mathbb{H}_3} |f(\mathbf{z})|^2 d^*\mathbf{z} < \infty$. A basis of differential operators Δ_1 and Δ_2 which commute with $GL(3)$ is given in [Gol06, p. 153], and are called Casimir operators.

Definition 1.6.1 (Maass form for $SL(3,\mathbb{Z})$). *Let $\nu = (\nu_1, \nu_2) \in \mathbb{C}^2$. A Maass form for $SL(3,\mathbb{Z})$ of type ν is a smooth function $f \in L^2(SL(3,\mathbb{Z})/\mathbb{H}_3)$ which satisfies*

1. *$f(\gamma \cdot z) = f(z)$ for all $\gamma \in SL(3,\mathbb{Z})$, $z \in \mathbb{H}_3$,*
2. *$\Delta_i f(z) = \lambda_i(\nu) f(z)$ for $i = 1, 2$,*
3. *for all $U_3(\mathbb{Z})$ upper triangular matrices with 1's in the diagonal entries,*

$$\int_{(SL(3,\mathbb{Z})\cap U_3(\mathbb{Z}))\backslash U_3(\mathbb{Z})} f(\boldsymbol{u} \cdot \boldsymbol{z}) d\boldsymbol{u} = 0.$$

Let $m = (m_1, m_2)$ with $m_1, m_2 \in \mathbb{Z}$ and $\nu = (\nu_1, \nu_2)$ with $\nu_1, \nu_2 \in \mathbb{C}$. The Jacquet-Whittaker function for $SL(3,\mathbb{Z})$ takes the form, for $\mathbf{z} \in \mathbb{H}_3$,

$$W_{Jacquet}(\mathbf{z}, \nu, \psi_m) := \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I_\nu(\omega_3 \cdot \mathbf{u} \cdot \mathbf{z})\overline{\psi_m(\mathbf{u})} du_{1,2} du_{1,3} du_{2,3},$$

where

$$\omega_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \mathbf{u} = \begin{pmatrix} 1 & u_{1,2} & u_{1,3} \\ 0 & 1 & u_{2,3} \\ 0 & 0 & 1 \end{pmatrix},$$

and

$$\psi_m(\mathbf{u}) = e^{2\pi i (m_1 u_{2,3} + m_2 u_{1,2})}, \quad I_\nu(\mathbf{z}) = y_1^{\nu_1 + 2\nu_2} y_2^{2\nu_1 + \nu_2}.$$

A Maass form for $SL(3,\mathbb{Z})$ can be written as the following Fourier series using the Jacquet-Whittaker functions,

$$\begin{aligned} f(\mathbf{z}) = &\sum_{\gamma \in U_2(\mathbb{Z})\backslash SL(2,\mathbb{Z})} \sum_{m_1=1}^{\infty} \sum_{m_2 \neq 0} \frac{A(m_1, m_2)}{|m_1 m_2|} \\ &\times W_{Jacquet}\left(\begin{pmatrix} |m_1 m_2| & 0 & 0 \\ 0 & m_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \cdot \mathbf{z},\ \nu,\ \psi_{1, \frac{m_2}{|m_2|}}\right), \end{aligned} \tag{1.29}$$

where $A(m_1, m_2) \in \mathbb{C}$ and $U_2(\mathbb{Z})$ are the upper triangular matrices with 1's in the diagonal entries.

The Fourier coefficients $A(m_1, m_2)$ satisfy

$$A(m_1, m_2) = A(m_1, -m_2)$$

for each $m_1 \geq 1$ and $m_2 \neq 0$.

As in the $SL(2,\mathbb{Z})$ case, we can define Hecke operators T_n on the space of Maass forms for $SL(3,\mathbb{Z})$. In particular, if we assume that f is a simultaneous eigenfunction of the Hecke operators for all n and we normalize the Maass form such that $A(1,1) = 1$, then

$$T_n f = A(n, 1) f,$$

and they satisfy the following multiplicative relations

$$A(m_1 m_1', m_2 m_2') = A(m_1, m_2) \cdot A(m_1', m_2') \ \text{ if } (m_1 m_2, m_1' m_2') = 1.$$

Definition 1.6.2 (Godement-Jacquet L-function). *Let $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 2$, and let f be a Maass form for $SL(3,\mathbb{Z})$, as in (1.29), normalized such that $A(1,1) = 1$ and which is an eigenfunction of all Hecke operators. The Godement-Jacquet L-function associated to f is* *defined as*

$$L(s,f) := \sum_{n=1}^{\infty} \frac{A(1,n)}{n^s} = \prod_p \left(1 - A(1,p)p^{-s} + A(p,1)p^{-2s} - p^{-3s}\right)^{-1}.$$

Theorem 1.6.1 (Analytic continuation and functional equation, [Gol06], p. 186)**.** *Let f be a Maass form of type $\nu = (\nu_1, \nu_2)$ for $SL(3,\mathbb{Z})$ with corresponding L-function $L(s,f)$. Then the L-function has a meromorphic continuation to all $s \in \mathbb{C}$ and satisfies the functional equation*

$$G_\nu(s)L(s,f) = \overline{G_\nu(1-s)}L(1-s,\bar{f}),$$

where

$$G_\nu(s) = \pi^{-3s/2}\Gamma\left(\frac{s+1-2\nu_1-\nu_2}{2}\right)\Gamma\left(\frac{s+\nu_1-\nu_2}{2}\right)\Gamma\left(\frac{s-1+\nu_1+2\nu_2}{2}\right).$$

If we then define

$$\begin{aligned} i\mu_1 &= 1 - 2\nu_1 - \nu_2, \\ i\mu_2 &= \nu_1 - \nu_2, \\ i\mu_3 &= -1 + \nu_1 + 2\nu_2, \end{aligned}$$

we obtain $\mu_1 + \mu_2 + \mu_3 = 0$, and the functional equation takes the compact form

$$\Lambda(s,f) := \Gamma_{\mathbb{R}}(s+i\mu_1)\Gamma_{\mathbb{R}}(s+i\mu_2)\Gamma_{\mathbb{R}}(s+i\mu_3)L(s,f) = \Lambda(1-s,\bar{f}).$$

Higher level $GL(3)$ Maass forms are defined similarly, with completed L-function having a factor of $N^{s/2}$ in front of it.

Examples of $GL(3)$ Maass forms of level 1 ($SL(3,\mathbb{Z})$ Maass forms) and 4 and their corresponding L-functions can be found in [FKL12] and they can be visualized more in detail in [LMF16, $GL(3)$ Maass forms].

$GL(d)$ Maass forms for $d > 3$ are defined analogously as in the three dimensional case. One starts with defining the generalized upper half-place as a quotient space, where each element can be written as a multiplication of two $d \times d$ matrices $\mathbf{x}$ and $\mathbf{y}$. One also introduces a $GL(d)$-invariant measure and proves that there are certain differential operators commuting with $GL(d)$. This procedure allows us to introduce Maass forms for $GL(d)$ that can be expressed as a Fourier series with a generalized Jacquet-Whittaker function as in (1.29).

For instance, an L-function $L(s,f)$ attached to a full level $SL(d,\mathbb{Z})$ Maass form has d $\Gamma_{\mathbb{R}}$-factors in its functional equation,

$$\Lambda(s,f) = \prod_{j=1}^{d} \Gamma_{\mathbb{R}}(s+i\mu_j)L(s,f),$$

where $\sum_j \mu_j = 0$. The functional equation satisfies Farmer's axioms (1.12) and thus the L-function belongs in the Selberg class.

Chapter 2

Methods and bounds for finding coefficients of L-functions

2.1 Introduction

In this chapter we introduce an improved method based on ideas put forward by Farmer, Koutsoliotas and Lemurell [FKL12], Booker [Boo06] and others to recover the initial Dirichlet coefficients of a primitive, self-dual L-function of any degree. The method is based on a new approximate functional equation which will be shown in the next section. The key advantage is that now the test-function used is allowed to be even or odd according to the specific L-function treated. This particular choice will make it a simple expression, easily implementable in computing programs.

Farmer, Koutsoliotas and Lemurell [FKL12] used Rubinstein's approximate functional equation (Theorem 1.3.2) to find primitive L-functions arising from $GL(3)$ and $GL(4)$ Maass forms. They also proved that in a certain spectral parameter region containing the origin, there can be no L-function, and thus no Maass forms with these eigenvalues. In particular, for the full level case $N = 1$, L-functions coming from $GL(d)$ Maass forms tend to have spectral parameters that are far (in norm) from the origin.

Instead, Booker [Boo06] used Rubinstein's approximate functional equation to locate the zeros of generic L-function and calculate the values of $L(s)$ along the critical line using the fast Fourier transform. This chapter will try to combine some ideas of Booker to the methods introduced by Farmer, Koutsoliotas and Lemurell in order to implement a method that efficiently recovers Dirichlet coefficients of an L-function given its Selberg data.

We may assume the following conjecture about the uniqueness of a primitive L-function in this chapter.

Conjecture 2.1.1. *Given Selberg data $(d, N, (\mu_j : \nu_k), \varepsilon)$, residue pair data at $s = 0$ and $s = 1$ of $\Lambda(s)$ and a primitive central character χ, there is at most one primitive L-function having these invariants (up to the reordering of the Gamma factors).*

The reason for assuming this conjecture is because the approximate functional equation uses deeply the functional equation (1.12) of an L-function and it is believed that these invariants, including the central character and the residues data, determine a primitive L-function uniquely.

In his PhD thesis, Molin [Mol10] explained in detail that a rigourous computation of L-functions using approximate functional equations is possible, and computed the complexity of Rubinstein's approximate functional equation in Theorem 1.3.2. His work motivates the implementation of an algorithm based on a new approximate functional equation that we will introduce in the next section.

For the rest of the chapter, we will assume for simplicity that the functional equation of a given L-function consists only of $\Gamma_{\mathbb{R}}$-terms, i.e. it is of the following form

$$\begin{aligned}\Lambda(s) &:= N^{s/2}\prod_{j=1}^{d}\Gamma_{\mathbb{R}}(s+\delta_j+i\mu_j)L(s) = N^{s/2}\gamma(s,\{\delta_j;\mu_j\})L(s) \\ &= \varepsilon\bar{\Lambda}(1-s), \end{aligned} \tag{2.1}$$

with $\delta_j \geq 0$ and μ_j real. This is possible without any loss of generality in Farmer's definition of the Selberg class, because the duplication formula holds,

$$\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1) = \Gamma_{\mathbb{C}}(s).$$

2.2 A new approximate functional equation

We will prove a more general form of the approximate functional equation that will allow to use a more sophisticated set of test-functions $g(s)$. In particular, by assuming additional conditions on $L(s)$ such as being self-dual, one can reduce the approximate functional equation to a single sum depending on the Dirichlet coefficients $b(n)$. The method is described in the following paragraph.

Consider $D \subset \mathbb{C}$, a subset of $\mathbb{C}$ such that $\mathbb{C}\setminus D$ is a set of isolated points and $g : D \to \mathbb{C}$ be a complex valued function such that

i) g is meromorphic and there exists a $\Sigma > 1/2$ such that $\Lambda(s)g(s)$ has no poles on the vertical lines $\mathrm{Re}(s) = \frac{1}{2} \pm \Sigma$,

ii) there exists a real positive sequence $(t_n)_{n\in\mathbb{N}}$ such that $t_n < t_{n+1}$ and $t_n \to +\infty$ with $|\Lambda(\sigma \pm it_n)g(\sigma \pm it_n)| \to 0$ on bounded strips $\sigma_1 \leq \sigma \leq \sigma_2$ as $n \to +\infty$. Moreover, on $\sigma_0 = \frac{1}{2} \pm \Sigma$ we have

$$\left|\int_{\sigma_0-i\infty}^{\sigma_0+i\infty} \Lambda(s)g(s)ds\right| < \infty,$$

that is, $\Lambda(s)g(s) \in L^1(\sigma_0 + i\mathbb{R})$.

Note that if $g(s)$ is holomorphic, then such a Σ exists for every value greater than $\frac{1}{2}$ and the second condition holds for a real positive variable t that goes to infinity.

The next theorem states the new approximate functional equation in its general form, where both $g(s)$ and $\Lambda(s)$ may have poles.

Theorem 2.2.1. *Let $g(s)$ be a complex valued function satisfying the conditions i) and ii) for a certain $\Sigma > \frac{1}{2}$ and let $L(s)$ be an L-function having Dirichlet coefficients $b(n)$ and functional equation of the form* (2.1). *Then*

$$\begin{aligned}\sum_{n=1}^{\infty} b(n)G_n(g(s),\Sigma,N,\{\delta_j;\mu_j\}) - \varepsilon\sum_{n=1}^{\infty}\overline{b(n)}G_n(g(1-s),\Sigma,N,\{\delta_j;-\mu_j\}) \\ = 2\pi i\sum_{s_j\in\Omega(\Sigma)}\operatorname*{Res}_{s=s_j}\Lambda(s)g(s), \end{aligned} \tag{2.2}$$

where

$$G_n(g(s),\Sigma,N,\{\delta_j;\mu_j\}) = \int_{(\Sigma+\frac{1}{2})}\gamma(s,\{\delta_j;\mu_j\})g(s)\left(\frac{n}{\sqrt{N}}\right)^{-s}ds,$$

$$\gamma(s, \{\delta_j; \mu_j\}) = \prod_{j=1}^{d} \Gamma_{\mathbb{R}}(s + \delta_j + i\mu_j),$$

and

$$\Omega(\Sigma) = \{s \in \mathbb{C} \ : \ |1/2 - \mathrm{Re}(s)| < \Sigma\}.$$

Proof. One has

$$\int_{\partial\Omega(\Sigma,T)} \Lambda(s)g(s)ds = 2\pi i \sum_{s_j \in \Omega(\Sigma,T)} \operatorname*{Res}_{s=s_j} \Lambda(s)g(s), \tag{2.3}$$

where $\partial\Omega(\Sigma, T)$ is the boundary of a rectangle centered at $\mathrm{Re}(s) = \frac{1}{2}$ with vertices at $(\frac{1}{2} \pm \Sigma \pm iT)$ for $\Sigma > \frac{1}{2}$ such that there is no pole of $g(s)$ on the contour line, and $\Omega(\Sigma, T)$ is the interior of the rectangle. In other words,

$$\Omega(\Sigma, T) = \{s \in \mathbb{C} \ : \ |1/2 - \mathrm{Re}(s)| < \Sigma, \ |\operatorname{Im}(s)| < T\}.$$

Consider now $T \to \infty$ through certain values of the sequence (t_n) described in *ii)*. Then, the equation (2.3) can be written as

$$\left(\int_{\frac{1}{2}+\Sigma-iT}^{\frac{1}{2}+\Sigma+iT} + \int_{\frac{1}{2}+\Sigma+iT}^{\frac{1}{2}-\Sigma+iT} + \int_{\frac{1}{2}-\Sigma+iT}^{\frac{1}{2}-\Sigma-iT} + \int_{\frac{1}{2}-\Sigma-iT}^{\frac{1}{2}+\Sigma-iT}\right) \Lambda(s)g(s)ds$$
$$= 2\pi i \sum_{s_j \in \Omega(\Sigma,T)} \operatorname*{Res}_{s=s_j} \Lambda(s)g(s),$$

where the second and the fourth integral tend to 0 by property *ii)*. Thus,

$$\left(\int_{\frac{1}{2}+\Sigma-i\infty}^{\frac{1}{2}+\Sigma+i\infty} - \int_{\frac{1}{2}-\Sigma-i\infty}^{\frac{1}{2}-\Sigma+i\infty}\right) \Lambda(s)g(s)ds = 2\pi i \sum_{s_j \in \Omega(\Sigma)} \operatorname*{Res}_{s=s_j} \Lambda(s)g(s), \tag{2.4}$$

where

$$\Omega(\Sigma) = \{s \in \mathbb{C} \ : \ |1/2 - \mathrm{Re}(s)| < \Sigma\}.$$

Because of *ii)*, both integrals in (2.4) are well-defined. For the first integral in (2.4) we have, using $L(s) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}$,

$$\sum_{n=1}^{\infty} b(n) \int_{\frac{1}{2}+\Sigma-i\infty}^{\frac{1}{2}+\Sigma+i\infty} N^{\frac{s}{2}} \prod_{j=1}^{d} \Gamma_{\mathbb{R}}(s + \delta_j + i\mu_j)g(s)n^{-s}ds.$$

For the second integral appearing in (2.4), we use the functional equation for $\Lambda(s)$ and a change of variables $s \leftrightarrow 1 - s$,

$$\int_{\frac{1}{2}-\Sigma-i\infty}^{\frac{1}{2}-\Sigma+i\infty} \Lambda(s)g(s)ds = \varepsilon \int_{\frac{1}{2}-\Sigma-i\infty}^{\frac{1}{2}-\Sigma+i\infty} \bar{\Lambda}(1-s)g(s)ds$$
$$= \varepsilon \sum_{n=1}^{\infty} \overline{b(n)} \int_{\frac{1}{2}-\Sigma-i\infty}^{\frac{1}{2}-\Sigma+i\infty} N^{\frac{1-s}{2}} \prod_{j=1}^{d} \Gamma_{\mathbb{R}}(1 - s + \delta_j - i\mu_j)g(s)n^{s-1}ds$$

$$= \varepsilon \sum_{n=1}^{\infty} \overline{b(n)} \int_{\frac{1}{2}+\Sigma-i\infty}^{\frac{1}{2}+\Sigma+i\infty} N^{\frac{s}{2}} \prod_{j=1}^{d} \Gamma_{\mathbb{R}}(s+\delta_j - i\mu_j) g(1-s) n^{-s} ds,$$

provided that $g(1-s)$ and the gamma factors have no poles on the line $\mathrm{Re}(s) = \Sigma + \frac{1}{2}$. □

The case where $g(s)$ and $\Lambda(s)$ are entire will be studied in this chapter and it's given in the following corollary.

Corollary 2.2.1. *Let $g(s)$ be a entire function satisfying the conditions i) and ii) for a certain $\Sigma > \frac{1}{2}$, and let $L(s)$ be an entire L-function with functional equation of the form (2.1). Then*

$$\sum_{n=1}^{\infty} b(n) G_n(g(s), \Sigma, N, \{\delta_j; \mu_j\}) - \varepsilon \sum_{n=1}^{\infty} \overline{b(n)} G_n(g(1-s), \Sigma, N, \{\delta_j; -\mu_j\}) = 0, \tag{2.5}$$

where G_n is defined in Theorem 2.2.1.

We considered the holomorphicity of $L(s)$ because primitive L-functions are assumed to be entire (except for the Riemann zeta-function) for any degree if the Selberg orthonormality conjecture (1.18) holds, see Theorem 1.2.2. Moreover, the assumption avoids considering the value of the residues of $\Lambda(s)$ at $s = 1$ and $s = 0$, which would increase the complexity of the numerical implementation.

2.2.1 Self-dual, odd and even test-functions

There are certain choices of the test-functions $g(s)$ that simplify equation (2.5) and make it simpler to compute numerically. For example, it is useful to separate the odd and the even parts of a test-function with respect to the critical line $\mathrm{Re}(s) = \frac{1}{2}$.

Definition 2.2.1. *A function is said to be even if $g(s) = g(1-s)$ and odd if $g(s) = -g(1-s)$.*

Form basic calculus we know that a function has an odd and even decompositon

$$g^e(s) = \frac{1}{2}(g(s) + g(1-s)) \tag{2.6}$$

and

$$g^o(s) = \frac{1}{2}(g(s) - g(1-s)), \tag{2.7}$$

so that $g^e(1-s) = g^e(s)$, $g^o(1-s) = -g^o(s)$ and $g^e(s) + g^o(s) = g(s)$. We call $g^o(s)$ the *odd part* of $g(s)$ and $g^e(s)$ the *even part* of $g(s)$.

The next theorem applies Theorem 2.2.1 to the unique odd and even decomposition of a test-function $g(s)$.

Theorem 2.2.2. *Let $L(s)$ be entire and let $g(s)$ be a holomorphic function satisfying i) and ii). Then their corresponding odd and even parts also satisfy i) and ii) for a common $\Sigma > 1/2$, and we have*

$$\begin{aligned}&\sum_{n=1}^{\infty} (b(n) G_n(g^e(s), \Sigma, N, \{\delta_j; \mu_j\}) - \epsilon \overline{b(n)} G_n(g^e(s), \Sigma, N, \{\delta_j; -\mu_j\})) \\ &+ \sum_{n=1}^{\infty} (b(n) G_n(g^o(s), \Sigma, N, \{\delta_j; \mu_j\}) + \epsilon \overline{b(n)} G_n(g^o(s), \Sigma, N, \{\delta_j; -\mu_j\})) = 0.\end{aligned} \tag{2.8}$$

Proof. The results follow by linearity of G_n applied to the even and odd decomposition of any test-function $g(s)$. $\square$

Theorem 2.2.2 tells us that, after an appropriate choice of the parity of the test-function, we can "kill" half of the terms of (2.8). For instance, if the test-function is even, the second sum in the LHS of equation (2.8) is canceled since the corresponding odd part is 0, and vice-versa in the case of odd test-functions.

In certain cases, a test-function which is invariant under conjugation simplifies the computations. A test-function $g(s)$ is called *self-dual* if

$$\bar{g}(s) := \overline{g(\bar{s})} = g(s). \tag{2.9}$$

Then $g(s)$ has equal magnitude with respect to the real axis, meaning that

$$|g(\sigma + it)| = |g(\sigma - it)|$$

for any real σ and t.

We can also prove the following lemma.

Lemma 2.2.1. *Assume that the test-function $g(s)$ is either totally even or odd and self-dual (2.9). Then the second Mellin transform G_n appearing in Theorem 2.2.2 is the (anti-)conjugate of the first one,*

$$G_n(g(1-s), \Sigma, N, \{\delta_j; -\mu_j\})) = (-1)^{\eta+1}\overline{G_n(g(s), \Sigma, N, \{\delta_j; \mu_j\}))},$$

where $\eta = 0$ if $g(s)$ is even and $\eta = 1$ otherwise. Moreover, if $L(s)$ is self-dual then

$$G_n(g(1-s), \Sigma, N, \{\delta_j; -\mu_j\})) = (-1)^{\eta}G_n(g(s), \Sigma, N, \{\delta_j; \mu_j\})),$$

and $G_n(g(s), \Sigma, N, \{\delta_j; \mu_j\}))$ is purely imaginary.

Proof. We have,

$$\begin{aligned}
\overline{G_n(g(1-s), \Sigma, N, \{\delta_j; -\mu_j\}))} &= \overline{\int_{(\Sigma+\frac{1}{2})} \gamma(s, \{\delta_j; -\mu_j\})g(1-s)\left(\frac{n}{\sqrt{N}}\right)^{-s} ds} \\
&= (-1)^{\eta}\int_{(\Sigma+\frac{1}{2})} \overline{\gamma(s, \{\delta_j; -\mu_j\})g(s)}\left(\frac{n}{\sqrt{N}}\right)^{-\bar{s}} \overline{ds} \\
&= (-1)^{\eta+1}\int_{(\Sigma+\frac{1}{2})} \gamma(\bar{s}, \{\delta_j; \mu_j\})g(\bar{s})\left(\frac{n}{\sqrt{N}}\right)^{-\bar{s}} ds.
\end{aligned}$$

The first statement follows after a change of variables $t \mapsto -t$. The second statement simply follows by noticing that, for self-dual L-functions, we have $\gamma(s, \{\delta_j; \mu_j\}) = \gamma(s, \{\delta_j; -\mu_j\})$. $\square$

If both $L(s)$ and $g(s)$ are self-dual, Theorem 2.2.2 simplifies even more.

Corollary 2.2.2. *If $L(s)$ and $g(s)$ are entire and self-dual, one of the following two cases holds.*

- *If $\varepsilon = 1$, then equation (2.8) reduces to*

$$\sum_{n=1}^{\infty} b(n)\,\mathrm{Im}(G_n(g^o(s), \Sigma, N, \{\delta_j; \mu_j\})) = 0. \tag{2.10}$$

- *If $\varepsilon = -1$, then equation (2.8) reduces to*

$$\sum_{n=1}^{\infty} b(n)\,\mathrm{Im}(G_n(g^e(s), \Sigma, N, \{\delta_j; \mu_j\})) = 0. \tag{2.11}$$

Remark 2.2.1. *If the L-function is self-dual, then the numerical implementation of the formulas (2.10) and (2.11) is straightforward, since they are directly dependent of the unknown $b(n)$, which are real-valued and the Mellin transforms G_n all purely imaginary. We can then rewrite our equation in the unknown $b(n)$ as*

$$\sum_{n=2}^{M} b(n)\,\mathrm{Im}(G_n(g(s), \Sigma, N, \{\delta_j; \mu_j\})) = -\,\mathrm{Im}(G_1(g(s), \Sigma, N, \{\delta_j; \mu_j\})), \tag{2.12}$$

with the test-function $g(s)$ depending on the value of the sign ε as shown in Corollary 2.2.2, with M to be chosen large enough to guarantee a desired precision of the equation. As long as M independent and valid set of test-functions are found, one can construct a matrix with entries consisting of the Mellin transforms of each test-function and different indexes n that can be used to find the coefficients $b(n)$ numerically.

Note that Remark 2.2.1 is essentially a different method to the one proposed in [FKL12]. Farmer et al. started with a simple linear combination of two fixed test-functions, and by evaluating Rubinstein's approximate functional equation at different points, they were able to generate new equations. In this method, a set of linear independent test-functions has to be found in order to apply Corollary 2.2.2 for the specific sign case.

In the next section, we will construct a specific set of test-functions that can be used for our implementation.

2.2.2 The choice of the test-function

For the explicit choice of the test-functions, one has to be really careful. In fact, one has observe the following facts.

1. For simplicity reasons, the test-functions should all be *entire* to avoid residue computations, as in the method developed in [FKL12].

2. The test-functions should be either all even or all odd for the cases shown in Corollary 2.2.2. The choice between an even or an odd set of test-functions should be implemented easily in its definition.

3. Similarly, it should be easy to switch between a self-dual and a non-self-dual set of test-functions, in order to apply Corollary 2.2.2 faithfully.

4. The test-functions should have a controlled exponential decrease along the vertical line $\mathrm{Re}(s) = \Sigma + \frac{1}{2}$, in order to simplify numerical computations. Moreover, the decrease of $\Gamma(s)$ along each vertical line should be taken into consideration, as suggested by Booker [Boo06].

5. The test-functions *must* be all independent when evaluated on the integration line $\mathrm{Re}(s) = \Sigma + \frac{1}{2}$, otherwise certain equations will be linear dependent and no solution can be computed. Thus, no linear combination of existing test-functions are allowed to generate new test-functions.

6. The more "orthogonal" the test-functions are with respect to the integral on the vertical line, the more the system of equations will be less ill-conditioned, and a more precise solution will be found. Therefore, a countable, quasi-orthogonal set of test-functions is preferable.

One can choose different possible sets of test-functions based on the observations of before. The following construction is used in our implementation for self-dual L-functions of any degree.

We can define, for the parameters $a, \tau \in \{0,1\}$, $k \in \mathbb{N}$, and $c \in \mathbb{C}$ the functions

$$g_{k,a,c,\tau}(s) = h_{k,a,c,\tau}\left(s - \frac{1}{2}\right), \tag{2.13}$$

where

$$h_{k,a,c,\tau}(s) = s^{2k+a} e^{cs^2} \psi_{d,\tau}(s), \tag{2.14}$$

and

$$\psi_{d,\tau}(s) = \begin{cases} 1 & , \text{ if } \tau = 0 \\ \cos\left(\frac{\pi s d}{4}\right) & , \text{ if } \tau = 1. \end{cases}$$

For any $k \in \mathbb{N}$, the functions are holomorphic, even if $a = 0$ and odd if $a = 1$. It is self-dual when c is a real number and non-self-dual when $\mathrm{Im}(c) \neq 0$.

The term $\cos\left(\frac{\pi s d}{4}\right)$ cancels the decrease of the gamma term along the vertical line because it is controlled for a fixed real part by

$$\left|\cos\left(\frac{\pi s d}{4}\right)\right| \asymp \exp\left(\frac{\pi d |t|}{4}\right).$$

It means that, up to irrelevant $|t|^\epsilon$ terms,

$$\left|g_{k,a,c,\tau}\left(\Sigma + \frac{1}{2} + it\right)\right| \asymp |t|^{2k+a} e^{-\mathrm{Re}(c)t^2 - \mathrm{Im}(c)t} \exp\left(\frac{\tau \pi d |t|}{4}\right)$$

so that

$$|\Lambda(s) g_{k,a,c,\tau}(s)| \asymp |t|^{2k+a} e^{-\mathrm{Re}(c)|t|^2 - \mathrm{Im}(c)t} \exp\left(\frac{(\tau - 1)\pi d |t|}{4}\right),$$

with the implied constant depending on Σ. For $\tau = 1$ and c real, the asymptotic of $\Lambda(s) g_{k,a,c,\tau}(s)$ looks like a Gaussian with a polynomial term t^{2k+a} in front of it. For $\tau = 0$ we will have a stronger decay due to the absence of the increase factor of the $\cos$ term.

The Hermite polynomials are known to be a set of orthogonal polynomials with respect to a Gaussian weight e^{-t^2}. Although the terms t^{2k+a} are different than the Hermite polynomials, our choice guarantees a quasi-orthogonal set of test-function along a fixed vertical line.

In Figure 2.1 we can see the effect of the odd test-function $g_{0,1,1,1}(s)$ on the decreasing rate of the Mellin transform G_n.

2.3 Description of the method

In the last section, we introduced a method to recover the Dirichlet coefficients of a self-dual, entire L-function given their Selberg data. We can now use this method to implement an algorithm that computes the first Dirichlet coefficients numerically.

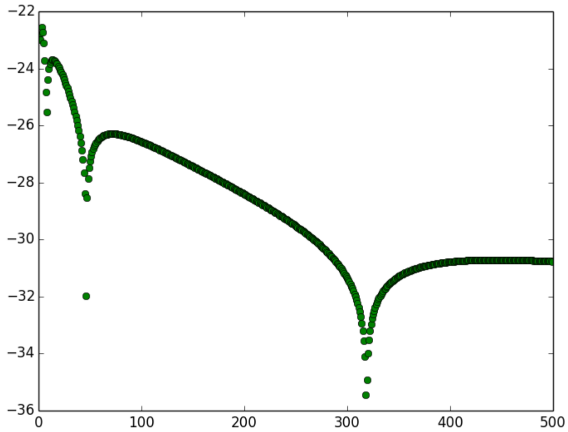

FIGURE 2.1: The norm of G_n in log-scale for $1 \leq n \leq 500$ in the case of the L-function associated to a level 1 $GL(4)$ Maass form with sign $\varepsilon = +1$, spectral parameters $(i \cdot 12.4687, i \cdot 4.72095, -i \cdot 12.4687, -i \cdot 4.72095)$ and the odd test-function $g_{0,1,1,1}(s) = (s-1/2)\exp((s-1/2)^2)\cos(\pi(s-1/2))$. The sudden drops are caused by the polynomial factor attached to the exponential in the test-function.

The algorithm starts with a fixed Selberg data $(d, N, (\delta_j + i\mu_j; 0), \varepsilon)$ and return its Dirichlet coefficients $b(n)$ using Remark 2.2.1 and the test-functions (2.13).

In order to provide an algorithm, we only need to specify the Mellin transforms G_n to high precision. We use the Riemann sum approximation of the corresponding integral,

$$R_n(m, T, g(s), \Sigma, N, \{\delta_j; \mu_j\}) = \sum_{j=1}^{m} \gamma(s_j, \{\delta_j, \mu_j\}) g(s_j) \left(\frac{n}{\sqrt{N}}\right)^{-s_j} \tag{2.15}$$

where $s_j = \Sigma + \frac{1}{2} + i(-T + \frac{2T}{m}j)$ runs along the vertical line with real part $\Sigma + \frac{1}{2}$, T is the truncation of the infinite integral in the range $[-T, T]$, and m is the number of steps in the Riemann sum. The approximated system then takes the form

$$\sum_{n=2}^{M} \tilde{b}(n) R_n(m, T, g(s), \Sigma, N, \{\delta_j; \mu_j\}) = -R_1(m, T, g(s), \Sigma, N, \{\delta_j; \mu_j\}), \tag{2.16}$$

which can be solved for $\tilde{b}(n)$ using a set of independent, self-dual, odd or even test-functions depending on the case treated.

2.3.1 Implementation in python

The programming language python was used to write the algorithm with the help of the package mpmath [Joh+14], a python library designed for computing with an arbitrary precision. One starts with the corresponding Selberg data $(d, N, (\delta_j + i\mu_j;), \varepsilon)$ of a supposedly genuine L-function.

The algorithm is then described in detail in the following Algorithm 1.

Algorithm 1: compute the initial Dirichlet coefficients of an L-function

Data: The Selberg data of a self-dual L-function $(d, N, (\delta_j + i\mu_j;), \varepsilon)$;
The number of steps m;
The truncation of the sum M;
The truncation of the infinite integral T;
$\Sigma > \frac{1}{2}$, $\tau \in \{0, 1\}$, $c > 0$;
Result: An approximation of the $M - 1$ Dirichlet coefficients $b(2), \ldots b(M)$;
Check parameters correctness;
if $\varepsilon = 1$ **then**
  Take odd test-functions $g_{k,1,c,\tau}(s)$ for $0 \leq k \leq M - 2$;
  Compute $R_n(m, T, g_{k,1,c,\tau}(s), \Sigma, N, \{\delta_j; \mu_j\})$ using (2.15) for any $0 \leq k \leq M - 2$, $1 \leq n \leq M$;
  Set up matrix **R** consisting of all $R_n(m, T, g_{k,1,c,\tau}(s), \Sigma, N, \{\delta_j; \mu_j\})$ for $0 \leq k \leq M - 2$, $2 \leq n \leq M$;
  Set up vector **c** consisting of $-R_1(m, T, g_{k,1,c,\tau}(s), \Sigma, N, \{\delta_j; \mu_j\})$ for $0 \leq k \leq M - 2$;
else
  Take even test-functions $g_{k,0,c,\tau}(s)$ for $0 \leq k \leq M - 2$;
  Compute $R_n(m, T, g_{k,0,c,\tau}(s), \Sigma, N, \{\delta_j; \mu_j\})$ using (2.15) for any $0 \leq k \leq M - 2$, $1 \leq n \leq M$;
  Set up matrix **R** consisting of all $R_n(m, T, g_{k,0,c,\tau}(s), \Sigma, N, \{\delta_j; \mu_j\})$ for $0 \leq k \leq M - 2$, $2 \leq n \leq M$;
  Set up vector **c** consisting of $-R_1(m, T, g_{k,0,c,\tau}(s), \Sigma, N, \{\delta_j; \mu_j\})$ for $0 \leq k \leq M - 2$;
Solve system $\mathbf{R} \cdot \mathbf{y} = \mathbf{c}$ for **y** using QR-factorization;
y approximates then $(b(2), \ldots, b(M))$.

The key part is choosing the parity of the test-functions according to the sign ε of the Selberg data of the L-function. The main disadvantage in the algorithm is that the constructed matrix **R** is in general ill-conditioned, and even small perturbations in the entries of the matrix could give relevant changes in the solution of the system. The QR-decomposition solves exactly the system according to the decomposition of **R** into a right-diagonal matrix and an orthogonal matrix. This is well suited for stability since the ortogonal matrix has the optimal condition number 1 and thus we don't loose precision by solving the othogonal part of the system.

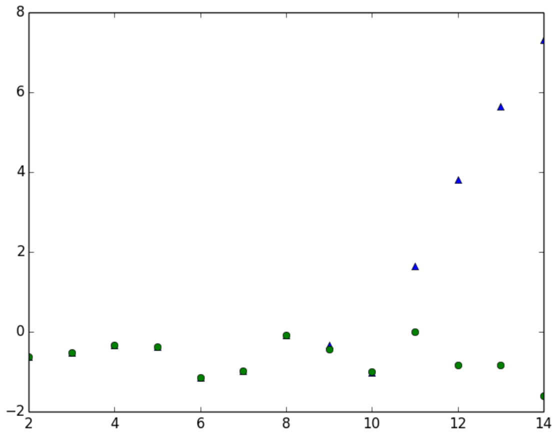

FIGURE 2.2: The absolute value of the Dirichlet coefficients $b(n)$ for $n = 2, 3, \ldots$ in log-scale, obtained through Algorithm 1 using the analytically normalized Ramanujan-Δ L-function (1.16). The computed solution is shown in blue, while the exact solution is shown in green.

Coefficients	Computed solution	Exact solution	Difference
$b(2)$	-0.53033008587	-0.53033008589	1.507×10^{-11}
$b(3)$	0.59873361178	0.598733612493	-7.041×10^{-10}
$b(4)$	-0.71874997314	-0.71875	2.685×10^{-8}
$b(5)$	0.69121251877	0.691213333205	-8.144×10^{-7}
$b(6)$	-0.31750697597	-0.317526448139	1.947×10^{-5}
$b(7)$	-0.37691733458	-0.376547696559	-3.696×10^{-4}
$b(8)$	0.9171482958	0.911504835123	0.005643460729
$b(9)$	-0.71179494787	-0.641518061271	-0.07027688660
$b(10)$	0.35679597726	-0.366571226367	0.7233672036
$b(11)$	-5.2275850477	1.0008729095	-6.2284579572
$b(12)$	44.904365262	-0.430339783979	45.334705046
$b(13)$	-281.91111907	-0.431561303293	-281.47955777
$b(14)$	1502.6725162	0.199694572258	1502.4728216

Figure 2.2 shows the results obtained with Algorithm 1 with the analytically normalized Ramanujan-Δ L-function (1.16). To generate the plot, the following parameters were chosen: $m = 5000$, $T = 100$, $\Sigma = 1$, $\tau = 0$, $c = 1$ and $M = 40$. The Dirichlet coefficients computed with Algorithm 1 and displayed in the table below show that for the initial coefficients there is a good approximation of the exact solution. Unfortunately, one can notice that the absolute error increases pretty rapidly with the increase of n, and the solution cannot be reliable anymore from a certain point because the Ramanujan hypothesis is violated (see Figure 2.2).

Several computations have shown that there is a very delicate balance between the parameters chosen and the precision of solution computed. To understand this connection better, we will study the estimate of the (relative) precision of the solution obtained in Algorithm 1 depending on the initial parameters.

2.4 Bounding the relative error of the solution of the algorithm

The interesting phenomenon appearing in Figure 2.2 motivates a more technical study of the behaviour of the solutions which are computed through Algorithm 1. We would like to estimate the relative error of the solution that we obtained in Algorithm 1, which is approximating the equation in an infinite number of variables

$$\sum_{n=2}^{\infty} b(n)G_n(g(s),\Sigma,N,\{\delta_j;\mu_j\}) = -G_1(g(s),\Sigma,N,\{\delta_j;\mu_j\})$$

with a finite sum of the form

$$\sum_{n=2}^{M} \tilde{b}(n)R_n(m,T,g(s),\Sigma,N,\{\delta_j;\mu_j\}) = -R_1(m,T,g(s),\Sigma,N,\{\delta_j;\mu_j\}),$$

with every R_n of the form

$$R_n(m,T,g(s),\Sigma,N,\{\delta_j;\mu_j\}) = \sum_{j=1}^{m} \gamma(s_j,\{\delta_j,\mu_j\})g(s_j)\left(\frac{n}{\sqrt{N}}\right)^{-s_j}, \tag{2.17}$$

where $s_j = \Sigma + \frac{1}{2} + i(-T + \frac{2T}{m}j)$ runs along the vertical line with real part $\Sigma + \frac{1}{2}$.

The *relative error* of a given solution $\mathbf{x}$ with respect to its approximated solution $\mathbf{y}$ is given by

$$\frac{||\mathbf{x}-\mathbf{y}||_v}{||\mathbf{x}||_v}, \tag{2.18}$$

where $||\cdot||_v$ is a norm which is compatible with a specific matrix norm $||\cdot||_m$ in the sense that

$$||\mathbf{B}\cdot\mathbf{z}||_v \le ||\mathbf{B}||_m \cdot ||\mathbf{z}||_v,$$

for any $n \times n$ matrix $\mathbf{B}$ and any n-dimensional vector $\mathbf{z}$. We will consider for simplicity the ∞-norm operator for vector and matrices. These are compatible norms, defined for a $k \times k$ matrix as

$$||\mathbf{B}||_\infty = \max_{1\le i\le k} \sum_{j=1}^{k} |b_{ij}|$$

and for vectors, as

$$||\mathbf{z}||_\infty = \max_{1\le j\le k} |z_j|.$$

In our specific case: $\mathbf{x} = (b(2),\ldots,b(M))$ and $\mathbf{y} = (\tilde{b}(2),\ldots,\tilde{b}(M))$, where the last vector is computed through Algorithm 1.

Let $(g_k)_{k\in\mathbb{N}}$ be a set of test-functions, and denote by $\mathbf{A} = (a_{k,n})_{k,n\in\mathbb{N}}$ with

$$a_{k,n} = G_n(g_k(s),\Sigma,N,\{\delta_j;\mu_j\})$$

and $\mathbf{b} = (b_k)_{k\in\mathbb{N}}$ with

$$b_k = -G_1(g_k(s),\Sigma,N,\{\delta_j;\mu_j\}),$$

Analogously, we can define the approximated system matrix $\mathbf{R} = (r_{k,n})_{1\le k\le M-1, 2\le n\le M}$, with

$$r_{k,n} = R_n(m,T,g_k(s),\Sigma,N,\{\delta_j;\mu_j\})$$

and the corresponding approximated vector $\mathbf{c} = (c_k)_{1\leq k\leq M-1}$, with

$$c_k = -R_1(m, T, g_k(s), \Sigma, N, \{\delta_j; \mu_j\}),$$

so that $\mathbf{R} \cdot \mathbf{y} = \mathbf{c}$.

The following theorem provides an upper bound for the relative error of the solution obtained in Algorithm 1.

Theorem 2.4.1. *The relative error (2.18) of the solution $\boldsymbol{y}$ of Algorithm 1 with respect to its exact solution $\boldsymbol{x}$ is bounded by*

$$\frac{||\boldsymbol{x}-\boldsymbol{y}||_\infty}{||\boldsymbol{x}||_\infty} \leq ||\boldsymbol{R}^{-1}||_\infty \left(C_1 \frac{\delta_2 + \delta_3}{||\boldsymbol{c}||_\infty - \delta_2} + \delta_4 \right),$$

whenever $||\boldsymbol{c}||_\infty > \delta_2$, where:

1. *C_1 is an upper absolute bound for*

$$\max_{1\leq k\leq M-1} \sum_{n=2}^{M} |G_n(g_k(s), \Sigma, N, \{\delta_j; \mu_j\})|;$$

2. *δ_2 is an upper absolute bound for*

$$\max_{1\leq k\leq M-1} |G_1(g_k(s), \Sigma, N, \{\delta_j; \mu_j\}) - R_1(m, T, g_k(s), \Sigma, N, \{\delta_j; \mu_j\})|;$$

3. *δ_3 is an upper absolute bound for the tail equation*

$$\max_{1\leq k\leq M-1} \left| \sum_{n=M+1}^{\infty} b(n) G_n(g_k(s), \Sigma, N, \{\delta_j; \mu_j\}) \right|;$$

4. *δ_4 is an upper absolute bound for the differences*

$$\max_{1\leq k\leq M-1} \sum_{n=2}^{M} |G_n(g_k(s), \Sigma, N, \{\delta_j; \mu_j\}) - R_n(m, T, g_k(s), \Sigma, N, \{\delta_j; \mu_j\})|.$$

Proof. Since $\mathbf{A}$ is an infinitely large matrix, one considers the truncation of it $\bar{\mathbf{A}}$ of the corresponding $M-1 \times M-1$ subsystem:

$$\sum_{n=2}^{M} b(n) G_n(g_k(s), \Sigma, N, \{\delta_j; \mu_j\}) = -G_1(g_k(s), \Sigma, N, \{\delta_j; \mu_j\}) \\ - \sum_{n=M+1}^{\infty} b(n) G_n(g_k(s), \Sigma, N, \{\delta_j; \mu_j\}),$$

in other words, $\bar{\mathbf{A}}\mathbf{x} = \bar{\mathbf{b}} + \mathbf{d}$, where the vector of dimension $M-1$ $\bar{\mathbf{b}}$ is equal to the first $M-1$ entries of $\mathbf{b}$, and $\mathbf{d}$ is the error of the remaining terms of the equations. Assuming that both $\bar{\mathbf{A}}$ and $\mathbf{R}$ are invertible matrices (it must be the case when the test-functions are all linear independent), then we would have

$$\mathbf{R}(\mathbf{x}-\mathbf{y}) = (\bar{\mathbf{b}} - \mathbf{c}) - (\bar{\mathbf{A}} - \mathbf{R})\mathbf{x} + \mathbf{d}$$

and thus for any vector norm $||\cdot||_v$ which is compatible with a matrix norm $||\cdot||_m$,

$$\begin{aligned}||\mathbf{x}-\mathbf{y}||_v &= ||\mathbf{R}^{-1}\left((\bar{\mathbf{b}}-\mathbf{c})-(\bar{\mathbf{A}}-\mathbf{R})\mathbf{x}+\mathbf{d}\right)||_v \\ &\leq ||\mathbf{R}^{-1}||_m\left(||\bar{\mathbf{b}}-\mathbf{c}||_v+||\bar{\mathbf{A}}-\mathbf{R}||_m\cdot||\mathbf{x}||_v+||\mathbf{d}||_v\right)\end{aligned}$$

We can now compute the relative error by dividing by $||\mathbf{x}||_v$:

$$\begin{aligned}\frac{||\mathbf{x}-\mathbf{y}||_v}{||\mathbf{x}||_v} &\leq ||\mathbf{R}^{-1}||_m\left(\frac{||\bar{\mathbf{b}}-\mathbf{c}||_v}{||\mathbf{x}||_v}+||\bar{\mathbf{A}}-\mathbf{R}||_m+\frac{||\mathbf{d}||_v}{||\mathbf{x}||_v}\right) \\ &\leq ||\mathbf{R}^{-1}||_m\left(||\bar{\mathbf{A}}||_m\frac{||\bar{\mathbf{b}}-\mathbf{c}||_v+||\mathbf{d}||_v}{||\bar{\mathbf{b}}+\mathbf{d}||_v}+||\bar{\mathbf{A}}-\mathbf{R}||_m\right) \\ &\leq ||\mathbf{R}^{-1}||_m\left(||\bar{\mathbf{A}}||_m\frac{||\bar{\mathbf{b}}-\mathbf{c}||_v+||\mathbf{d}||_v}{||\bar{\mathbf{b}}||_v}+||\bar{\mathbf{A}}-\mathbf{R}||_m\right)\end{aligned}$$

One can then write $\mathbf{c}=\mathbf{c}-\bar{\mathbf{b}}+\bar{\mathbf{b}}$ so that

$$||\bar{\mathbf{b}}||_v \geq ||\mathbf{c}||_v - ||\mathbf{c}-\bar{\mathbf{b}}||_v.$$

The vector $||\mathbf{c}||_v$ can be computed exactly and the upper bound for $||\mathbf{c}-\bar{\mathbf{b}}||_v$ gives the lower bound for $||\mathbf{c}||_v-||\mathbf{c}-\bar{\mathbf{b}}||_v$. As long as this last term is positive, it provides a valid estimate for the whole formula.

Finally, if we take the ∞-norm and we assume that $||\bar{\mathbf{A}}||_\infty \leq C_1$, $||\bar{\mathbf{b}}-\mathbf{c}||_\infty \leq \delta_2$, $||\mathbf{d}||_\infty \leq \delta_3$ and $||\bar{\mathbf{A}}-\mathbf{R}||_\infty \leq \delta_4$, and assuming that $||\mathbf{c}||_\infty - \delta_2 > 0$, then the relative error is bounded by

$$\frac{||\mathbf{x}-\mathbf{y}||_\infty}{||\mathbf{x}||_\infty} \leq ||\mathbf{R}^{-1}||_\infty\left(C_1\frac{\delta_2+\delta_3}{||\mathbf{c}||_\infty-\delta_2}+\delta_4\right),$$

and the theorem is proved. □

In the next sections, we will provide upper bounds in the particular cases appearing in Theorem 2.4.1.

1. The upper bound for $||\bar{\mathbf{A}}||_\infty$ as well as the upper bound for $||\mathbf{d}||_\infty$ will be treated in §2.4.1.
2. The upper bound for $||\bar{\mathbf{b}}-\mathbf{c}||_\infty$ as well as $||\bar{\mathbf{A}}-\mathbf{R}||_\infty$ will be considered in §2.4.2.

2.4.1 Bounding G_n and the tail equation

For simplicity, we define new constants that allow us to simplify computations appearing in the following sections, such as $\mathfrak{c}=\frac{\Sigma}{2}+\frac{1}{4}$, $\mathfrak{a}_j=\mathfrak{c}+\frac{\delta_j}{2}$, $\mathfrak{b}=\sum_j \mathfrak{a}_j$ and $\mathfrak{d}=d+1$.

The following lemma can be proved easily with the Riemann-Lebesgue lemma.

Lemma 2.4.1. *Suppose that $g(s)$ satisfies the properties i) and ii) in section 2.2. Then the integral expression $G_n(g(s),\Sigma,N,\{\delta_j;\mu_j\})$ in Theorem 2.2.1 satisfies the following bound as $n\to+\infty$:*

$$G_n(g(s),\Sigma,N,\{\delta_j;\mu_j\}) = o\left(\left(\frac{n}{\sqrt{N}}\right)^{-\Sigma-\frac{1}{2}}\right).$$

Proof. We have

$$G_n(g(s),\Sigma,N,\{\delta_j;\mu_j\})$$

$$= \int_{(\Sigma+\frac{1}{2})} \gamma(s, \{\delta_j;\mu_j\}) g(s) \left(\frac{n}{\sqrt{N}}\right)^{-s} ds$$
$$= i \left(\frac{n}{\sqrt{N}}\right)^{-\Sigma-\frac{1}{2}} \int_{-\infty}^{\infty} \gamma(\Sigma+1/2+it, \{\delta_j;\mu_j\}) g(\Sigma+1/2+it) \left(\frac{n}{\sqrt{N}}\right)^{-it} dt.$$

The last integral can be viewed as a suitably normalized Fourier transform. By the Riemann-Lebesgue lemma, since $\gamma(s, \{\delta_j;\mu_j\})g(s)$ is summable on $\mathrm{Re}(s) = \Sigma + \frac{1}{2}$, the whole integral tends to 0 as $n \to +\infty$. $\square$

The next lemma provides a more specific upper bound, where the implied constant is exactly determined. We used in particular the Hölder inequality to separate each Γ-factor and the test function $g(s)$.

Lemma 2.4.2. *Let* $\mathfrak{c} = \frac{\Sigma}{2} + \frac{1}{4}$, $\mathfrak{a}_j = \mathfrak{c} + \frac{\delta_j}{2}$, $\mathfrak{b} = \sum_j \mathfrak{a}_j$, $\mathfrak{d} = d+1$ *and*

$$K = N^{\mathfrak{c}} \pi^{-2\mathfrak{b}+d-\frac{d}{\mathfrak{d}}} 2^{\frac{3d}{\mathfrak{d}}+\mathfrak{b}} e^{\pi\mathfrak{b}/2} \mathfrak{d}^{-\mathfrak{b}+d/2-\frac{d}{\mathfrak{d}}} \prod_{j=1}^{d} \Gamma\left(\mathfrak{d}(\mathfrak{a}_j - 1/2)+1\right)^{\frac{1}{\mathfrak{d}}}.$$

If $g(s) \in L^{\mathfrak{d}}\left(\Sigma + \frac{1}{2} + i\mathbb{R}\right)$*, then*

$$|G_n(g(s), \Sigma, N, \{\delta_j;\mu_j\})| \leq n^{-\Sigma-\frac{1}{2}} K ||g(s)||_{L^{\mathfrak{d}}\left(\Sigma+\frac{1}{2}+i\mathbb{R}\right)}, \tag{2.19}$$

where for any $x \in \mathbb{R}$ *and any* $p \geq 1$*,*

$$||g(s)||_{L^p(x+i\mathbb{R})} = \left(\int_{-\infty}^{\infty} |g(x+it)|^p \, dt\right)^{\frac{1}{p}}.$$

Proof. We have

$$\begin{aligned}
&|G_n(g(s), \Sigma, N, \{\delta_j;\mu_j\})| \\
&= \left| \int_{\mathrm{Re}(s)=2\mathfrak{c}} \gamma(s, \{\delta_j;\mu_j\}) g(s) \left(\frac{n}{\sqrt{N}}\right)^{-s} ds \right| \\
&= \left| \int_{-\infty}^{\infty} N^{\mathfrak{c}+it/2} \prod_{j=1}^{d} \Gamma\left(\mathfrak{a}_j + i\frac{\mu_j+t}{2}\right) \pi^{-(\mathfrak{a}_j+i(\mu_j+t)/2)} g(2\mathfrak{c}+it) n^{-2\mathfrak{c}-it} i \, dt \right| \\
&\leq N^{\mathfrak{c}} \pi^{-\mathfrak{b}} n^{-2\mathfrak{c}} \int_{-\infty}^{\infty} |g(2\mathfrak{c}+it)| \prod_{j=1}^{d} \left| \Gamma\left(\mathfrak{a}_j + i\frac{\mu_j+t}{2}\right) \right| dt.
\end{aligned}$$

Consider now each single integral inside the product only. Applying the Hölder inequality in each of the $\mathfrak{d} = d+1$ terms, we obtain:

$$\begin{aligned}
&\int_{-\infty}^{\infty} |g(2\mathfrak{c}+it)| \prod_{j=1}^{d} \left| \Gamma\left(\mathfrak{a}_j + i\frac{\mu_j+t}{2}\right) \right| dt \\
&\leq \left(\int_{-\infty}^{\infty} |g(2\mathfrak{c}+it)|^{\mathfrak{d}} \, dt\right)^{\frac{1}{\mathfrak{d}}} \prod_{j=1}^{d} \left(\int_{-\infty}^{\infty} \left| \Gamma\left(\mathfrak{a}_j + i\frac{\mu_j+t}{2}\right) \right|^{\mathfrak{d}} dt\right)^{\frac{1}{\mathfrak{d}}}
\end{aligned}$$

$$\leq ||g(s)||_{L^{\mathfrak{d}}(2\mathfrak{c}+i\mathbb{R})} \prod_{j=1}^{d} \left(\int_{-\infty}^{\infty} \left| \Gamma \left(\mathfrak{a}_j + i\frac{\mu_j + t}{2} \right) \right|^{\mathfrak{d}} dt \right)^{\frac{1}{\mathfrak{d}}}. \tag{2.20}$$

Now, note that from the Stirling formula,

$$|\Gamma(\mathfrak{a}_j + it)| \leq (2\pi)^{1/2} (\mathfrak{a}_j + |t|)^{\mathfrak{a}_j - 1/2} e^{-\pi|t|/2}, \quad t \in \mathbb{R}.$$

Hence,

$$\begin{aligned}
\int_{-\infty}^{\infty} |\Gamma(\mathfrak{a}_j + it)|^{\mathfrak{d}} \, dt &\leq 2(2\pi)^{\mathfrak{d}/2} \int_0^{\infty} (\mathfrak{a}_j + t)^{\mathfrak{d}(\mathfrak{a}_j - 1/2)} e^{-\pi \mathfrak{d} t/2} dt \\
&= 2(2\pi)^{\mathfrak{d}/2} e^{\pi \mathfrak{d} \mathfrak{a}_j/2} \int_{\mathfrak{a}_j}^{\infty} t^{\mathfrak{d}(\mathfrak{a}_j - 1/2)} e^{-\pi \mathfrak{d} t/2} dt \\
&\leq 2(2\pi)^{\mathfrak{d}/2} e^{\pi \mathfrak{d} \mathfrak{a}_j/2} \int_0^{\infty} t^{\mathfrak{d}(\mathfrak{a}_j - 1/2)} e^{-\pi \mathfrak{d} t/2} dt \\
&= 2(2\pi)^{\mathfrak{d}/2} e^{\pi \mathfrak{d} \mathfrak{a}_j/2} \left(\frac{2}{\mathfrak{d}\pi} \right)^{\mathfrak{d}(\mathfrak{a}_j - 1/2) + 1} \int_0^{\infty} t^{\mathfrak{d}(\mathfrak{a}_j - 1/2)} e^{-t} dt \\
&= 2(2\pi)^{\mathfrak{d}/2} e^{\pi \mathfrak{d} \mathfrak{a}_j/2} \left(\frac{2}{\mathfrak{d}\pi} \right)^{\mathfrak{d}(\mathfrak{a}_j - 1/2) + 1} \Gamma\left(\mathfrak{d}(\mathfrak{a}_j - 1/2) + 1 \right)
\end{aligned} \tag{2.21}$$

where $\mathfrak{d} = d + 1$. With this, and continuing the bound (2.20), we get for an appropriate change of variable $\tilde{t} = (\mu_j + t)/2$ for each integral

$$\begin{aligned}
&\prod_{j=1}^{d} \left(\int_{-\infty}^{\infty} \left| \Gamma \left(\mathfrak{a}_j + i\frac{\mu_j + t}{2} \right) \right|^{\mathfrak{d}} dt \right)^{\frac{1}{\mathfrak{d}}} \\
&= \prod_{j=1}^{d} \left(2 \int_{-\infty}^{\infty} |\Gamma(\mathfrak{a}_j + it)|^{\mathfrak{d}} \, dt \right)^{\frac{1}{\mathfrak{d}}} \\
&\leq \prod_{j=1}^{d} \left(4(2\pi)^{\mathfrak{d}/2} e^{\pi \mathfrak{d} \mathfrak{a}_j/2} \left(\frac{2}{\mathfrak{d}\pi} \right)^{\mathfrak{d}(\mathfrak{a}_j - 1/2) + 1} \Gamma\left(\mathfrak{d}(\mathfrak{a}_j - 1/2) + 1 \right) \right)^{\frac{1}{\mathfrak{d}}} \\
&= 4^{\frac{d}{\mathfrak{d}}} (2\pi)^{d/2} e^{\pi \mathfrak{b}/2} \left(\frac{2}{\mathfrak{d}\pi} \right)^{(\mathfrak{b} - d/2) + \frac{d}{\mathfrak{d}}} \prod_{j=1}^{d} \Gamma\left(\mathfrak{d}(\mathfrak{a}_j - 1/2) + 1 \right)^{\frac{1}{\mathfrak{d}}}.
\end{aligned}$$

Collecting all the terms, we obtain

$$\begin{aligned}
&|G_n(g(s), \Sigma, N, \{\delta_j; \mu_j\})| \\
&\leq n^{-2\mathfrak{c}} N^{\mathfrak{c}} \pi^{-2\mathfrak{b} + d - \frac{d}{\mathfrak{d}}} 2^{\frac{3d}{\mathfrak{d}} + \mathfrak{b}} e^{\pi \mathfrak{b}/2} \mathfrak{d}^{-\mathfrak{b} + d/2 - \frac{d}{\mathfrak{d}}} \\
&\quad \times \prod_{j=1}^{d} \left(\mathfrak{d}(\mathfrak{a}_j - 1/2) + 1 \right)^{\frac{1}{\mathfrak{d}}} ||g(s)||_{L^{\mathfrak{d}}(2\mathfrak{c}+i\mathbb{R})} \\
&= n^{-2\mathfrak{c}} K ||g(s)||_{L^{\mathfrak{d}}(2\mathfrak{c}+i\mathbb{R})}.
\end{aligned}$$

□

With Lemma 2.4.2 and assuming the Ramanujan hypothesis (1.11), we can bound the tail equation.

Lemma 2.4.3. *Let $\mathfrak{c}, \mathfrak{a}_j, \mathfrak{b}, \mathfrak{d}, K$ as in Lemma 2.4.2. Assume the Ramanujan hypothesis $|b(n)| \leq Cn^{\epsilon}$ for $C = C(\epsilon) > 0$ and $\epsilon > 0$, and suppose that $\epsilon < 2\mathfrak{c} - 1 = \Sigma - \frac{1}{2}$. Then for any $M \in \mathbb{N}_{\geq 1}$ we have*

$$\left| \sum_{n>M} b(n) G_n(g(s), \Sigma, N, \{\delta_j; \mu_j\}) \right| \leq K_1 ||g(s)||_{L^{\mathfrak{d}}(\Sigma + \frac{1}{2} + i\mathbb{R})}, \tag{2.22}$$

where

$$K_1 = \frac{CK}{2\mathfrak{c} - \epsilon - 1} M^{-2\mathfrak{c} + \epsilon + 1}.$$

Proof. By Lemma 2.4.2,

$$\left| \sum_{n>M} b(n) G_n(g(s), \Sigma, N, \{\delta_j; \mu_j\}) \right| \leq CK ||g(s)||_{L^{\mathfrak{d}}(2\mathfrak{c} + i\mathbb{R})} \sum_{n>M} n^{-2\mathfrak{c} + \epsilon}.$$

Now,

$$\sum_{n>M} n^{-2\mathfrak{c}+\epsilon} \leq \int_M^{\infty} x^{-2\mathfrak{c}+\epsilon} dx = -\frac{1}{-2\mathfrak{c} + \epsilon + 1} M^{-2\mathfrak{c}+\epsilon+1},$$

which is convergent if $-2\mathfrak{c} + \epsilon + 1 < 0$, i.e. $\epsilon < 2\mathfrak{c} - 1$; thus

$$\left| \sum_{n>M} b(n) G_n(g(s), \Sigma, N, \{\delta_j; \mu_j\}) \right| \leq K_1 ||g(s)||_{L^{\mathfrak{d}}(2\mathfrak{c} + i\mathbb{R})}$$

with the implied constant

$$K_1 = \frac{CK}{2\mathfrak{c} - \epsilon - 1} M^{-2\mathfrak{c} + \epsilon + 1}.$$

□

Note that, to ensure the condition $0 < \epsilon < \Sigma - \frac{1}{2}$, one can increase the real part of the integration line Σ.

Remark 2.4.1. *To choose the truncation of the sum such that the precision of a single equation is at least ψ, one needs to solve the inequality from Lemma 2.4.3 and obtain*

$$M > \exp\left(\frac{1}{1 + \epsilon - 2\mathfrak{c}} \log\left(\frac{(2\mathfrak{c} - \epsilon - 1)\psi}{CK\mathfrak{M}} \right) \right) \tag{2.23}$$

where $\mathfrak{M} = ||g(s)||_{L^{\mathfrak{d}}(2\mathfrak{c} + i\mathbb{R})}$.

2.4.2 Bounding the error arising from the approximations of G_n

The remaining term that needs to be bounded in Theorem 2.4.1 is the error coming from the approximation of the Mellin transform G_n. We would like to approximate the Mellin transform

$$G_n(g(s), \Sigma, N, \{\delta_j; \mu_j\}) = \int_{(\Sigma + \frac{1}{2})} \gamma(s, \{\delta_j, \mu_j\}) g(s) \left(\frac{n}{\sqrt{N}} \right)^{-s} ds$$

by a Riemann sum of the type

$$R_n(m,T,\Sigma,N,\{\delta_j;\mu_j\}) = \sum_{j=1}^{m} \gamma(s_j,\{\delta_j,\mu_j\})g(s_j)\left(\frac{n}{\sqrt{N}}\right)^{-s_j},$$

where $s_j = \Sigma + \frac{1}{2} + i(-T + \frac{2T}{m}j)$. We need to compute the error of the approximation depending on T, m and the test function that we are currently using. We will use the same notation as before: $2\mathfrak{c} = \Sigma + \frac{1}{2}$, $\mathfrak{a}_j = a + \frac{\delta_j}{2}$ and $\mathfrak{b} = \sum_j \alpha_j$ and $\mathfrak{d} = d+1$.

We have

$$\begin{aligned}
&|G_n(g(s),\Sigma,N,\{\delta_j;\mu_j\}) - R_n(m,T,\Sigma,N,\{\delta_j;\mu_j\})| \\
&\leq \left|G_n(g(s),\Sigma,N,\{\delta_j;\mu_j\}) - \int_{\substack{(\Sigma+\frac{1}{2})\\ |\operatorname{Im}(s)|\leq T}} \gamma(s,\{\delta_j,\mu_j\})g(s)\left(\frac{n}{\sqrt{N}}\right)^{-s} ds\right| \\
&\quad + \left|\int_{\substack{(\Sigma+\frac{1}{2})\\ |\operatorname{Im}(s)|\leq T}} \gamma(s,\{\delta_j,\mu_j\})g(s)\left(\frac{n}{\sqrt{N}}\right)^{-s} ds - R_n(m,T,\Sigma,N,\{\delta_j;\mu_j\})\right| \\
&=: \Psi_{1,n} + \Psi_{2,n}.
\end{aligned} \tag{2.24}$$

The terms $\Psi_{1,n}$ and $\Psi_{2,n}$ need to be treated in more detail.

2.4.2.1 Bounding $\Psi_{1,n}$

The first term $\Psi_{1,n}$ is bounded by using again Hölder's inequality.

Lemma 2.4.4. *Assume that the test-function $g(s)$ is self-dual. The first term $\Psi_{1,n}$ in (2.24) can be bounded as*

$$\begin{aligned}
\Psi_{1,n} \leq\, & 2n^{-2\mathfrak{c}}N^{\mathfrak{c}}\pi^{-2\mathfrak{b}+d-\frac{d}{\mathfrak{d}}}2^{\frac{3d}{\mathfrak{d}}+\mathfrak{b}}e^{\pi\mathfrak{b}/2}\mathfrak{d}^{-\mathfrak{b}+d/2-\frac{d}{\mathfrak{d}}} \\
&\times \Gamma\left(\mathfrak{d}\left(\mathfrak{c}+\frac{\max_j\delta_j - 1}{2}\right)+1, \frac{2}{\mathfrak{d}\pi}(T+\mathfrak{c})\right)^{\frac{d}{\mathfrak{d}}} ||g(s)||_{L^{\mathfrak{d}}(2\mathfrak{c}+i\mathbb{R})},
\end{aligned}$$

where $\Gamma(s,x)$ is the upper incomplete Gamma function.

Proof. We have

$$\begin{aligned}
\Psi_{1,n} &= \left|G_n(g(s),\Sigma,N,\{\delta_j;\mu_j\}) - \int_{\substack{(\Sigma+\frac{1}{2})\\ |\operatorname{Im}(s)|\leq T}} \gamma(s,\{\delta_j,\mu_j\})g(s)\left(\frac{n}{\sqrt{N}}\right)^{-s} ds\right| \\
&= \left|\left(\int_{-\infty}^{-T} + \int_{T}^{+\infty}\right)\gamma(2\mathfrak{c}+it,\{\delta_j,\mu_j\})g(2\mathfrak{c}+it)\left(\frac{n}{\sqrt{N}}\right)^{-2\mathfrak{c}-it} dt\right|.
\end{aligned}$$

Assuming that $g(s)$ is self-dual, we have equal magnitude with respect to the real axis, therefore,

$$\Psi_{1,n} \leq 2\left(\frac{n}{\sqrt{N}}\right)^{-2\mathfrak{c}} \int_T^{\infty} |\gamma(2\mathfrak{c}+it,\{\delta_j,\mu_j\})g(2\mathfrak{c}+it)|\, dt$$

$$= 2\left(\frac{n}{\sqrt{N}}\right)^{-2\mathfrak{c}} \int_T^\infty \left| \prod_{j=1}^d \Gamma\left(\mathfrak{a}_j + i\frac{\mu_j + t}{2}\right) \pi^{-\mathfrak{a}_j - i(t+\mu_j)/2} g(2\mathfrak{c} + it) \right| dt$$
$$\leq 2\left(\frac{n}{\sqrt{N}}\right)^{-2\mathfrak{c}} \pi^{-\mathfrak{b}} \int_T^\infty \prod_{j=1}^d \left| \Gamma\left(\mathfrak{a}_j + i\frac{\mu_j + t}{2}\right) g(2\mathfrak{c} + it) \right| dt.$$

And again applying Hölder's inequality to the $d+1$ terms. Furthermore, using the same techniques as in (2.21), we obtain

$$\begin{aligned}
&\int_T^\infty |\Gamma(\mathfrak{a}_j + it)|^{\mathfrak{d}}\, dt \\
&\leq 2(2\pi)^{\mathfrak{d}/2} \int_T^\infty (\mathfrak{a}_j + t)^{\mathfrak{d}(\mathfrak{a}_j - 1/2)} e^{-\pi\mathfrak{d}t/2} dt \\
&= 2(2\pi)^{\mathfrak{d}/2} e^{\pi\mathfrak{d}\mathfrak{a}_j/2} \int_{T+\mathfrak{a}_j}^\infty t^{\mathfrak{d}(\mathfrak{a}_j - 1/2)} e^{-\pi\mathfrak{d}t/2} dt \\
&= 2(2\pi)^{\mathfrak{d}/2} e^{\pi\mathfrak{d}\mathfrak{a}_j/2} \left(\frac{2}{\mathfrak{d}\pi}\right)^{\mathfrak{d}(\mathfrak{a}_j - 1/2)+1} \int_{\frac{2}{\mathfrak{d}\pi}(T+\mathfrak{a}_j)}^\infty t^{\mathfrak{d}(\mathfrak{a}_j - 1/2)} e^{-t} dt \\
&= 2(2\pi)^{\mathfrak{d}/2} e^{\pi\mathfrak{d}\mathfrak{a}_j/2} \left(\frac{2}{\mathfrak{d}\pi}\right)^{\mathfrak{d}(\mathfrak{a}_j - 1/2)+1} \Gamma\left(\mathfrak{d}(\mathfrak{a}_j - 1/2) + 1, \frac{2}{\mathfrak{d}\pi}(T + \mathfrak{a}_j)\right),
\end{aligned}$$

where $\Gamma(s, x)$ is the upper incomplete Gamma function, and therefore

$$\begin{aligned}
\Psi_{1,n} &\leq 2\left(\frac{n}{\sqrt{N}}\right)^{-2\mathfrak{c}} \pi^{-\mathfrak{b}} \prod_{j=1}^d \left(\int_T^\infty \left|\Gamma\left(\mathfrak{a}_j + i\frac{\mu_j + t}{2}\right)\right|^{\mathfrak{d}} dt\right)^{\frac{1}{\mathfrak{d}}} ||g(s)||_{L^{\mathfrak{d}}(2\mathfrak{c}+i\mathbb{R})} \\
&= 2\left(\frac{n}{\sqrt{N}}\right)^{-2\mathfrak{c}} \pi^{-\mathfrak{b}} 2^{\frac{d}{\mathfrak{d}}} \prod_{j=1}^d \left(\int_T^\infty |\Gamma(\mathfrak{a}_j + it)|^{\mathfrak{d}} dt\right)^{\frac{1}{\mathfrak{d}}} ||g(s)||_{L^{\mathfrak{d}}(2\mathfrak{c}+i\mathbb{R})} \\
&\leq 2\left(\frac{n}{\sqrt{N}}\right)^{-2\mathfrak{c}} \pi^{-\mathfrak{b}} 4^{\frac{d}{\mathfrak{d}}} (2\pi)^{d/2} e^{\pi\mathfrak{b}/2} \left(\frac{2}{\mathfrak{d}\pi}\right)^{(\mathfrak{b}-d/2)+\frac{d}{\mathfrak{d}}} \\
&\quad \times \prod_{j=1}^d \Gamma\left(\mathfrak{d}(\mathfrak{a}_j - 1/2) + 1, \frac{2}{\mathfrak{d}\pi}(T + \mathfrak{a}_j)\right)^{\frac{1}{\mathfrak{d}}} ||g(s)||_{L^{\mathfrak{d}}(2\mathfrak{c}+i\mathbb{R})} \\
&\leq 2\left(\frac{n}{\sqrt{N}}\right)^{-2\mathfrak{c}} \pi^{-\mathfrak{b}} 4^{\frac{d}{\mathfrak{d}}} (2\pi)^{d/2} e^{\pi\mathfrak{b}/2} \left(\frac{2}{\mathfrak{d}\pi}\right)^{(\mathfrak{b}-d/2)+\frac{d}{\mathfrak{d}}} \\
&\quad \times \Gamma\left(\mathfrak{d}\left(\mathfrak{c} + \frac{\max_j \delta_j - 1}{2}\right) + 1, \frac{2}{\mathfrak{d}\pi}(T + \mathfrak{c})\right)^{\frac{d}{\mathfrak{d}}} ||g(s)||_{L^{\mathfrak{d}}(2\mathfrak{c}+i\mathbb{R})}.
\end{aligned}$$

This last term could be evaluated once we know the $L^{\mathfrak{d}}$ norm of our test-function. Collecting the terms altogether as in Lemma 2.4.2 we have the result. Notice that the whole term goes to 0 as $T \to +\infty$ because of the incomplete Gamma function term. □

2.4.2.2 Bounding $\Psi_{2,n}$

For the second quantity $\Psi_{2,n}$, one can use the standard error arising from the Riemann sum approximation.

Lemma 2.4.5. *We have*

$$\Psi_{2,n} \leq \frac{2T^2}{m}\left(\frac{n}{\sqrt{N}}\right)^{-2\mathfrak{c}} \pi^{-\mathfrak{b}}(2\pi)^{d/2}\exp\left(-\frac{\pi}{4}\sum_j |\mu_j|\right)\prod_{j=1}^{d}\left(\mathfrak{a}_j + \frac{1}{2}(T+\max_j|\mu_j|)\right)^{\mathfrak{a}_j-\frac{1}{2}}$$
$$\left(\frac{1-\log(\pi)}{2}\frac{d}{a} + d\gamma + \frac{\pi^2}{6}\left(\mathfrak{b} + \frac{d}{2}(T+\max_j|\mu_j|)\right)\mathfrak{g} + \mathfrak{g}_1 + \left|\log\left(\frac{n}{\sqrt{N}}\right)\right|\mathfrak{g}\right),$$

where $\mathfrak{g} = \mathfrak{g}(T,\mathfrak{c}) = \max_{t\in[-T,T]}|g(2\mathfrak{c}+it)|$ *and* $\mathfrak{g}_1 = \mathfrak{g}_1(T,\mathfrak{c}) = \max_{t\in[-T,T]}|g'(2\mathfrak{c}+it)|$.

Proof. The error arising from the Riemann sum approximation is bounded by

$$\begin{aligned}\Psi_{2,n} &\leq \frac{2T^2}{m}\left(\frac{n}{\sqrt{N}}\right)^{-2\mathfrak{c}} \cdot \max_{t\in[-T,T]}\left|\frac{d}{dt}\gamma(2\mathfrak{c}+it,\{\delta_j,\mu_j\})g(2\mathfrak{c}+it)\left(\frac{n}{\sqrt{N}}\right)^{-it}\right| \\ &= \frac{2T^2}{m}\left(\frac{n}{\sqrt{N}}\right)^{-2\mathfrak{c}} \cdot \max_{t\in[-T,T]}|\mathfrak{M}(t)|. \end{aligned}\tag{2.25}$$

The quantity $\mathfrak{M}(t)$ can be explicitly evaluated,

$$\begin{aligned}&\mathfrak{M}(t)\\ &= \sum_{k=1}^{d}\prod_{\substack{1\leq j\leq d\\ j\neq k}}\Gamma_{\mathbb{R}}(2\mathfrak{c}+\delta_j+i(t+\mu_j))\left(\frac{d}{dt}\Gamma_{\mathbb{R}}(2\mathfrak{c}+\delta_k+i(t+\mu_k))\right)g(2\mathfrak{c}+it)\left(\frac{n}{\sqrt{N}}\right)^{-it}\\ &+\gamma(2\mathfrak{c}+it,\{\delta_j,\mu_j\})\left(\frac{d}{dt}g(2\mathfrak{c}+it)\right)\left(\frac{n}{\sqrt{N}}\right)^{-it}\\ &+\gamma(2\mathfrak{c}+it,\{\delta_j,\mu_j\})g(2\mathfrak{c}+it)\left(-i\log\left(\frac{n}{\sqrt{N}}\right)\right)\left(\frac{n}{\sqrt{N}}\right)^{-it}.\end{aligned}$$

For the derivatives of the $\Gamma_{\mathbb{R}}$-terms, we have

$$\begin{aligned}&\frac{d}{dt}\Gamma_{\mathbb{R}}(2\mathfrak{a}_k+i(t+\mu_k))\\ &=\frac{d}{dt}\left(\pi^{-\mathfrak{a}_k-i(t+\mu_k)/2}\Gamma\left(\mathfrak{a}_k+\frac{i(t+\mu_k)}{2}\right)\right)\\ &=-i\frac{\log(\pi)}{2}\Gamma_{\mathbb{R}}(2\mathfrak{a}_k+i(t+\mu_k))+\frac{i}{2}\pi^{-\mathfrak{a}_k-i(t+\mu_k)/2}\Gamma'(s)|_{s=\mathfrak{a}_k+\frac{i(t+\mu_k)}{2}},\end{aligned}$$

where $\mathfrak{a}_k = \mathfrak{c} + \frac{\delta_k}{2}$. Now, we can use the relation

$$\Gamma'(s) = \frac{d}{ds}\Gamma(s) = \Gamma(s)\psi_0(s)$$

where ψ_0 is the polygamma function of order zero, defined as

$$\psi_0(s) = \frac{\Gamma'}{\Gamma}(s),$$

which can be expressed explicitly using Weierstrass factorization formula, as

$$\psi_0(s) = -\frac{1}{s} - \gamma - \sum_{n=1}^{\infty}\left(\frac{1}{n+s}-\frac{1}{n}\right), \quad s\neq 0,-1,-2,-3,\ldots \tag{2.26}$$

where $\gamma = -\Gamma'(1)$ is the Euler-Mascheroni constant, so that

$$\frac{d}{dt}\Gamma_{\mathbb{R}}(2\mathfrak{a}_k + i(t+\mu_k)) = i\frac{1-\log(\pi)}{2}\Gamma_{\mathbb{R}}(2\mathfrak{a}_k + i(t+\mu_k))\psi_0\left(\mathfrak{a}_k + \frac{i(t+\mu_k)}{2}\right) \tag{2.27}$$

We need to bound ψ_0 using (2.26) along vertical lines $s = \sigma + it$, where $|t| \to \infty$,

$$\begin{aligned}\psi_0(\sigma+it) &= -\frac{1}{\sigma+it} - \gamma - \sum_{n=1}^{\infty}\left(\frac{1}{n+\sigma+it} - \frac{1}{n}\right)\\ &= -\frac{\sigma-it}{\sigma^2+t^2} - \gamma - \sum_{n=1}^{\infty}\frac{n-(n+\sigma+it)}{(n+\sigma+it)n}\\ &= -\frac{\sigma-it}{\sigma^2+t^2} - \gamma + (\sigma+it)\sum_{n=1}^{\infty}\frac{1}{(n+\sigma+it)n}.\end{aligned}$$

Thus,

$$\begin{aligned}|\psi_0(\sigma+it)| &\le \frac{1}{|\sigma+it|} + \gamma + |\sigma+it|\left|\sum_{n=1}^{\infty}\frac{1}{(n+\sigma+it)n}\right|\\ &\le \frac{1}{|\sigma+it|} + \gamma + \frac{\pi^2}{6}|\sigma+it| \end{aligned}\tag{2.28}$$

because

$$\left|\sum_{n=1}^{\infty}\frac{1}{(n+\sigma+it)n}\right| \le \sum_{n=1}^{\infty}\frac{1}{n(n+\sigma)} \le \sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

Inserting (2.28) and (2.27) into $\mathfrak{M}(t)$:

$$\begin{aligned}&\mathfrak{M}(t)\\ &= \sum_{k=1}^{d}\prod_{\substack{1\le j\le d\\ j\ne k}}\Gamma_{\mathbb{R}}(2\mathfrak{c}+\delta_j+i(t+\mu_j))\left(\frac{d}{dt}\Gamma_{\mathbb{R}}(2\mathfrak{c}+\delta_k+i(t+\mu_k))\right)g(2\mathfrak{c}+it)\left(\frac{n}{\sqrt{N}}\right)^{-it}\\ &\quad+\gamma(2\mathfrak{c}+it,\{\delta_j,\mu_j\})\left(\frac{d}{dt}g(2\mathfrak{c}+it)\right)\left(\frac{n}{\sqrt{N}}\right)^{-it}\\ &\quad+\gamma(2\mathfrak{c}+it,\{\delta_j,\mu_j\})g(2\mathfrak{c}+it)\left(-i\log\left(\frac{n}{\sqrt{N}}\right)\right)\left(\frac{n}{\sqrt{N}}\right)^{-it}\\ &= i\frac{1-\log(\pi)}{2}\gamma(2\mathfrak{c}+it,\{\delta_j,\mu_j\})g(2\mathfrak{c}+it)\left(\frac{n}{\sqrt{N}}\right)^{-it}\sum_{k=1}^{d}\psi_0\left(\mathfrak{a}_k+\frac{i(t+\mu_k)}{2}\right)\\ &\quad+\gamma(2\mathfrak{c}+it,\{\delta_j,\mu_j\})\left(\frac{d}{dt}g(2\mathfrak{c}+it)\right)\left(\frac{n}{\sqrt{N}}\right)^{-it}\\ &\quad+\gamma(2\mathfrak{c}+it,\{\delta_j,\mu_j\})g(2\mathfrak{c}+it)\left(-i\log\left(\frac{n}{\sqrt{N}}\right)\right)\left(\frac{n}{\sqrt{N}}\right)^{-it}\\ &= \gamma(2\mathfrak{c}+it,\{\delta_j,\mu_j\})\left(\frac{n}{\sqrt{N}}\right)^{-it}\left(i\frac{1-\log(\pi)}{2}g(2\mathfrak{c}+it)\sum_{k=1}^{d}\psi_0\left(\mathfrak{a}_k+\frac{i(t+\mu_k)}{2}\right)\right.\\ &\quad\left.+ig'(s)|_{s=2\mathfrak{c}+it} - i\log\left(\frac{n}{\sqrt{N}}\right)g(2\mathfrak{c}+it)\right),\end{aligned}$$

so that for any $t \in [-T, T]$,

$$
\begin{aligned}
|\mathfrak{M}(t)| \leq |\gamma(2\mathfrak{c}+it, \{\delta_j, \mu_j\})| & \\
\left(\frac{1-\log(\pi)}{2}|g(2\mathfrak{c}+it)|\left|\sum_{k=1}^{d}\psi_0\left(\mathfrak{a}_k+\frac{i(t+\mu_k)}{2}\right)\right|\right. & \\
\left.+\left|g'(2\mathfrak{c}+it)\right|+\left|\log\left(\frac{n}{\sqrt{N}}\right)\right||g(2\mathfrak{c}+it)|\right). &
\end{aligned}
$$

Now we can bound the first term,

$$
\begin{aligned}
|\Gamma_{\mathbb{R}}(2\mathfrak{c}+\delta_j+i(t+\mu_j))| &\leq \pi^{-\mathfrak{a}_j}\left|\Gamma\left(\mathfrak{a}_j+i\frac{t+\mu_j}{2}\right)\right| \\
&\leq \pi^{-\mathfrak{a}_j}\sqrt{2\pi}\left(\mathfrak{a}_j+\left|\frac{t+\mu_j}{2}\right|\right)^{\mathfrak{a}_j-\frac{1}{2}} e^{-\frac{\pi}{4}|t+\mu_j|}
\end{aligned}
$$

and thus,

$$
\begin{aligned}
|\gamma(2\mathfrak{c}+it, \{\delta_j, \mu_j\})| &\leq \pi^{-\mathfrak{b}}\prod_{j=1}^{d}\left|\Gamma\left(\mathfrak{a}_j+i\frac{t+\mu_j}{2}\right)\right| \\
&\leq \pi^{-\mathfrak{b}}(2\pi)^{d/2}\exp\left(-\frac{\pi}{4}\sum_j|t+\mu_j|\right)\prod_{j=1}^{d}\left(\mathfrak{a}_j+\frac{1}{2}|t+\mu_j|\right)^{\mathfrak{a}_j-\frac{1}{2}} \\
&\leq \pi^{-\mathfrak{b}}(2\pi)^{d/2}\exp\left(-\frac{\pi}{4}\sum_j|\mu_j|\right)\prod_{j=1}^{d}\left(\mathfrak{a}_j+\frac{1}{2}(T+\max_j|\mu_j|)\right)^{\mathfrak{a}_j-\frac{1}{2}}.
\end{aligned}
$$

For the sum of the polygamma terms, because of (2.28), we have

$$
\left|\sum_{k=1}^{d}\psi_0\left(\mathfrak{a}_k+\frac{i(t+\mu_k)}{2}\right)\right| \leq \frac{d}{a}+d\gamma+\frac{\pi^2}{6}\left(\mathfrak{b}+\frac{d}{2}(T+\max_j|\mu_j|)\right),
$$

and thus we obtain,

$$
\begin{aligned}
|\mathfrak{M}(t)| \leq \pi^{-\mathfrak{b}}(2\pi)^{d/2}\exp\left(-\frac{\pi}{4}\sum_j|\mu_j|\right)\prod_{j=1}^{d}\left(\mathfrak{a}_j+\frac{1}{2}(T+\max_j|\mu_j|)\right)^{\mathfrak{a}_j-\frac{1}{2}} & \\
\left(\frac{1-\log(\pi)}{2}\frac{d}{a}+d\gamma+\frac{\pi^2}{6}\left(\mathfrak{b}+\frac{d}{2}(T+\max_j|\mu_j|)\right)\mathfrak{g}+\mathfrak{g}_1+\left|\log\left(\frac{n}{\sqrt{N}}\right)\right|\mathfrak{g}\right). &
\end{aligned}
\tag{2.29}
$$

The lemma then follows by inserting (2.29) in (2.25). □

2.5 Conclusions and future work

Due to unexpected complications in the numerical implementation of the algorithm, I was not able to develop this method any further during my PhD. However, I believe that it was worth publishing inside my thesis because I the results I obtained could lead to further possible fascinating developments and be used by other mathematicians to study this problem.

In particular, interesting future work would be using Theorem 2.4.1 to define Θ as being an upper bound for the absolute error of the solution of Algorithm 1:

$$\max_j |x_j - y_j| \leq \max_j |x_j| \cdot ||\mathbf{R}^{-1}|| \left(C_1 \frac{\delta_2 + \delta_3}{||\mathbf{c}||_\infty - \delta_2} + \delta_4 \right)$$
$$=: \Theta(m, T, M, \Sigma, N, \{\delta_j; \mu_j\}),$$

and set up an optimization problem of the following form

$$\begin{aligned} \min\ & \Theta(m, T, M, \Sigma, N, \{\delta_j; \mu_j\}) \text{ such that} \\ & ||\mathbf{c}||_\infty > \delta_2 \\ & \mathcal{R} \leq \tau, \end{aligned}$$

where the running time $\mathcal{R}$ of the Algorithm 1 doesn't exceed a certain time τ. Optimizing this quantity would provide an optimal choice for the parameters involved in Algorithm 1 in order to recover the desired Dirichlet coefficients to an absolute precision of at most Θ. This would hopefully clarify the curious phenomenon appeared in Figure 2.2.

Chapter 3

The largest gap between zeros of general L-functions is less than 41.54

3.1 Introduction

This chapter is based on a preprint [KRZ], in a collaboration with N. Robles and A. Zaharescu.

The Grand Riemann Hypothesis (Conjecture 1.2.2) states that the zeros of any L-function are on the critical line $\mathrm{Re}(s) = \frac{1}{2}$. We may then list the zeros as $\frac{1}{2} + i\gamma_n$ for $n \in \mathbb{Z} \setminus \{0\}$, where

$$\cdots < \gamma_{-2} < \gamma_{-1} \leq 0 < \gamma_1 < \gamma_2 < \cdots$$

and we refer to $\frac{1}{2} + i\gamma_1$, or simply γ_1, as the "first" zero of an L-function.

Miller [Mil02] originally studied the location of the first zero for automorphic L-functions of a real archimedean type. He observed that L-functions with small real spectral parameters and conductor do not usually have a small first zero compared to the height of the first zero of the Riemann zeta-function. He noticed that using Weil's explicit formula on the GRH there is at least a zero of every such L-functions in the interval $[-\gamma_{1,\zeta}, \gamma_{1,\zeta}]$, where $\gamma_{1,\zeta} \approx 14.13472$ is the imaginary part of the first zero of the Riemann zeta-function.

Things seem to behave differently if we allow the spectral parameter to be a complex number. In [BCF+15], Bober, Conrey, Farmer, Fujii, Koutsoliotas, Lemurell, Rubinstein and Yoshida exhibited a particular degree 4 L-function coming from a $GL(4)$ Maass form with first zero greater than the first zero of $\zeta(s)$. A possible motivation for this surprising discovery is that the imaginary part of the first trivial zeros of the corresponding L-function appear close to a minimum of the (Hardy) Z-function,

$$Z(t) = e^{i\theta(t)} L\left(\frac{1}{2} + it\right),$$

where $\theta(t)$ is such that $Z(t)$ is real. These particular trivial zeros seem to suppress the appearance of a nearby zero, thereby forcing the first zero to have a higher imaginary part. Miller was also able to show the same result for $GL(2)$ Maass forms of any level.

Because of the counterexample provided for a specific L-function, the question they raised was: "does there exist another higher absolute upper bound for the highest lowest zero of general L-functions, including L-functions coming from $GL(d)$ Maass forms?" In the same paper [BCF+15] they proved an upper bound of 45.3236 for the length of the interval containing at least one zero, which is unfortunately much worse than Miller's result of $2 \cdot \gamma_{1,\zeta} \approx 28.26944$ for L-functions of real archimedean type. This new upper bound is obtained using a different set of functions introduced by Selberg

[Sel91] whose properties are described in [CCM15]. We will call them Selberg's functions.

In this chapter we improve a striking result of [BCF+15] regarding the upper bound of the largest gap between zeros of general entire L-functions from 45.3236 to 41.54 using suitable feasible pairs and convex combinations of Selberg minorant functions under GRH and Ramanujan hypothesis.

Bober [BCF+15] suggested that the lowest upper bound should be around 36. However, no indication is given of how this number is obtained. A future paper of Bober [Bob] mentioned in [BCF+15] will clarify this point.

Based on the properties of feasible sets (see Definition 3.4.1 below), one arrives at a more detailed upper bound. Furthermore, a natural question that arises would be to see whether this bound is optimal, in the sense that either there would exist at least one L-function with exactly this largest gap, or it would be an accumulation point of differences between imaginary parts of zeros of general L-functions.

At the end of the chapter, another useful application of Selberg's functions is provided. Assuming GRH and the Ramanujan hypothesis for the first Dirichlet coefficient $b(2)$, no entire L-function with completed L-function of the form

$$Q^s\Gamma_{\mathbb{R}}(s)^d L(s),$$

where $Q^2 = N$ is the conductor of the L-function according to Farmer's definition and defined in (1.12), exists for certain values of Q too small (to be quantified in §3.5). Here $\Gamma_{\mathbb{R}}(s)$ is defined in (1.13). In particular, for degree $d = 4$, there is no entire L-function of this form having conductor $N < 324$, while for $d = 5$ there are no such L-functions having $N < 1375$.

3.1.1 The largest gap between zeros of an L-function

We denote the entire L-function subclass by $\mathcal{S}_{\text{hol}}$.

Definition 3.1.1 (Largest gap). *Assume GRH. For each $L \in \mathcal{S}_{\text{hol}}$ let Gap_L be the largest gap between consecutive zeros of $L(s)$. By the largest gap between zeros of general L-functions we mean the quantity*

$$\operatorname{Gap}_* := \sup_{L\in\mathcal{S}_{\text{hol}}} \operatorname{Gap}_L .$$

We will use the following notation for the logarithmic derivative of a L-function,

$$-\frac{L'}{L}(s) = \sum_{n=1}^{\infty} \frac{\Lambda_L(n)}{n^s}, \tag{3.1}$$

where $\Lambda_L(n)$ is the generalized von Mangoldt function for L-functions. Note that since $L(s)$ has an Euler product, it follows that $\Lambda_L(n)$ is supported on prime powers only, and is given by

$$\Lambda_L(n) = \begin{cases} 0, & \text{if } n = 1, n \neq p^k, \\ \log p \sum_{j=1}^{d'} \alpha_{j,p}^k, & \text{if } n = p^k, k \geq 1, \end{cases} \tag{3.2}$$

where $b(n)$ are the Dirichlet coefficients and $\alpha_{j,p}$ the Satake parameters associated to the local Euler factor at p. If p is good prime $p \nmid N$, then $d' = d$, if it is a bad prime $p|N$, then $d' < d$. In particular, the corresponding factor is $\Lambda_L(p) = b(p)\log p$ at primes p, while at $n = p^2$ this is equal to $(2b(p^2) - b(p)^2)\log p$.

The axioms of L-functions predict the validity of Ramanujan's hypothesis, although it has only been proved for a limited subclass of L-functions. For instance, the hypothesis (1.11) is still unproved for L-functions coming from Maass forms.

Since in this chapter we need to bound Dirichlet coefficients of L-functions, for the sake of completeness we give results assuming both conditional as well as unconditional bounds (on the result of the Luo-Rudnick-Sarnak, Theorem 1.4.2).

One can bound the Dirichlet coefficients of the logarithmic derivative of an L-function conditionally and unconditionally as follows.

If k is a positive integer, then at all good primes $p \nmid N$:

$$|\Lambda_L(n)| \leq \begin{cases} d \log p, & \text{on the Ramanujan hypothesis, if } n = p^k, \\ dp^{\frac{k}{2} - \frac{k}{d^2+1}} \log p, & \text{on the Luo-Rudnick-Sarnak bound, if } n = p^k, \\ 2p^{7/64} \log p, & \text{on the Kim-Sarnak bound for degrees } d \leq 2, n = p^k. \end{cases} \tag{3.3}$$

The main result of the chapter is the following.

Theorem 3.1.1. *Assume GRH, then the gap between any two consecutive non-trivial zeros of an entire L-function is less than 41.54 on the Ramanujan hypothesis, and it is less than 43.41 on the Luo-Rudnick-Sarnak bound. In other words,*

$$\mathrm{Gap}_* \leq \begin{cases} 41.54, & \textit{on the Ramanujan hypothesis,} \\ 43.41, & \textit{on the Luo-Rudnick-Sarnak bound.} \end{cases}$$

Under the Ramanujan hypothesis the interval is reduced from 45.324 to under 41.54, i.e. nearly a 8.3% improvement from the previous result. Unfortunately, improvements are much harder to find for weaker bound on the coefficients, but a small improvement from the upper bound of 45.3236 is nonetheless provided.

3.2 Weil explicit formula

The analytic tool used is the Weil explicit formula, which we will apply to Selberg's functions defined in the next section. As in the beginning of Chapter 2, we will assume that the L-function has a functional equation consisting of $\Gamma_{\mathbb{R}}$-terms only (see (2.1)). Therefore, in our setting the completed L-function takes the form

$$Q^s \gamma(s, \{\mu_j\}) L(s) = Q^s \prod_{j=1}^{d} \Gamma_{\mathbb{R}}(s + \mu_j) L(s) \tag{3.4}$$

where $Q = \sqrt{N}$, $\mathrm{Re}(\mu_j) \geq 0$ and $\sum_j \mathrm{Im}(\mu_j) = 0$ (according to Farmer's definition).

The explicit formula is the following.

Theorem 3.2.1. *Assume GRH. Suppose that $L(s)$ has a Dirichlet series expansion (1.10) which continues to an entire function with functional equation having (3.4) as Γ-factors, and suppose that*

$$L(\sigma + it) \ll |t|^A,$$

for $A > 0$ uniformly in t and bounded σ. Let $f(s)$ be holomorphic in a horizontal strip $-(1/2+\delta) < \mathrm{Im}(s) < 1/2 + \delta$ with $f(s) \ll \min(1, |s|^{-(1+\epsilon)})$ in this region and suppose that $f(x)$ is *real valued for real x. Suppose that the Fourier transform of f defined by*

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(u)e^{-2\pi iux}du$$

is such that

$$\sum_{n=1}^{\infty}\left(\frac{-\Lambda_L(n)}{\sqrt{n}}\hat{f}\left(\frac{\log n}{2\pi}\right) + \frac{-\bar{\Lambda}_L(n)}{\sqrt{n}}\hat{f}\left(-\frac{\log n}{2\pi}\right)\right)$$

is absolutely convergent. Then we have

$$\sum_{\gamma} f(\gamma) = \frac{\hat{f}(0)}{\pi}\log Q + \frac{1}{2\pi}\sum_{j=1}^{d}\ell(\mu_j, f)$$
$$+ \frac{1}{2\pi}\sum_{n=1}^{\infty}\left(\frac{-\Lambda_L(n)}{\sqrt{n}}\hat{f}\left(\frac{\log n}{2\pi}\right) + \frac{-\bar{\Lambda}_L(n)}{\sqrt{n}}\hat{f}\left(-\frac{\log n}{2\pi}\right)\right), \tag{3.5}$$

where the sum $\sum_{\gamma}$ runs over the non-trivial zeros of $L(s)$, and

$$\ell(\mu, f) = \mathrm{Re}\left(\int_{-\infty}^{\infty}\frac{\Gamma'}{\Gamma}\left(\frac{1}{4}+\frac{it}{2}+\mu\right)f(t)dt\right) - \hat{f}(0)\log\pi. \tag{3.6}$$

Proof. This can be found in Iwaniec and Kowalski [IK04, Page 109], but using a different normalization of the Fourier transform. □

Following the remarkably clever idea put forward in [BCF+15], we apply Weil's formula to a specific entire function f such that it approximates the characteristic function $\chi_{[\alpha,\beta]}(x)$ for x real, and such that that the support of $\hat{f}$ is compact and located within a certain region. The last term of (3.5) would then be a finite sum that can be bounded using conditional or unconditional bounds on the Dirichlet coefficients.

3.3 Beurling function and Selberg's functions

This section and relative results are summarized from [CCM15, §2] and [BCF+15, §4]. In the 1930s, Beurling studied the real entire function

$$B(z) = 1 + 2\left(\frac{\sin(\pi z)}{\pi}\right)^2\left(\frac{1}{z} - \sum_{n=1}^{\infty}\frac{1}{(n+z)^2}\right).$$

This function is a good smooth approximation of the sign function

$$\mathrm{sgn}(x) = \begin{cases} -1, & \text{if } x < 0,\\ 0, & \text{if } x = 0,\\ 1, & \text{if } x > 0,\end{cases}$$

and it is a majorant of $\mathrm{sgn}(x)$, meaning that $\mathrm{sgn}(x) \le B(x)$ for all $x \in \mathbb{R}$. For all real x, the Fourier transform $\hat{B}(x)$ is supported on $[-1, 1]$ and satisfies

$$\int_{-\infty}^{\infty}|B(x) - \mathrm{sgn}(x)|\,dx = 1.$$

Moreover, Beurling showed that $B(x)$ minimized the $L^1(\mathbb{R})$-distance to $\operatorname{sgn}(x)$. Selberg used the Beurling function to define two entire functions in the following way.

Definition 3.3.1 (Selberg minorant and majorant functions)**.** *For a parameter $\delta > 0$, the Selberg minorant and majorant functions are defined in the interval $[\alpha, \beta]$ to be*

$$S^-(z) = S^-_{\alpha,\beta;\delta}(z) := -\frac{1}{2}\left(B(\delta(\alpha - z)) + B(\delta(z-\beta))\right), \tag{3.7}$$

and

$$S^+(z) = S^+_{\alpha,\beta;\delta}(z) := \frac{1}{2}\left(B(\delta(-\alpha + z)) + B(\delta(\beta - z))\right). \tag{3.8}$$

Selberg observed that $S^-(x) \le \chi_{[\alpha,\beta]}(x) \le S^+(x)$ for all real x and that their Fourier transforms have support in $[-\delta, \delta]$, as shown in Figure 3.1.

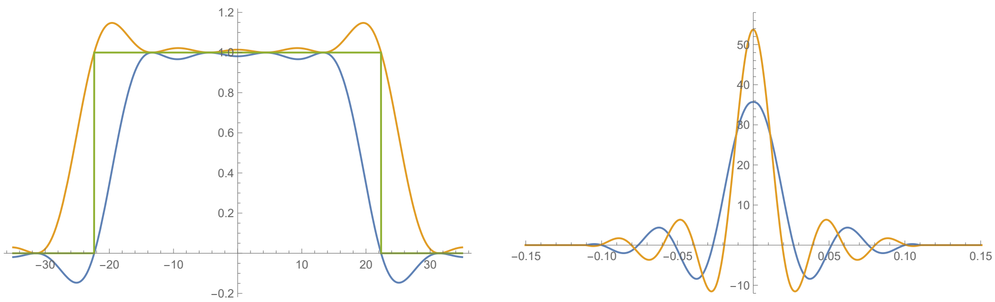

FIGURE 3.1: On the left hand side: the Selberg minorant and majorant functions in blue and orange respectively, with $\beta = 22.36$, $\delta = 2.5/\beta$. The green line is the characteristic function of the interval $[-\beta, \beta]$. The second graph shows their Fourier transforms.

The next lemma summarizes the main properties of $S^\pm(z)$.

Lemma 3.3.1 (Properties of $S^\pm(z)$)**.** *Let $S^\pm(z)$ be the Selberg minorant/majorant functions in the interval $[\alpha, \beta]$ and parameter $\delta > 0$. One has*

1. $S^-(x) \le \chi_{[\alpha,\beta]}(x) \le S^+(x)$ *for all real* x.
2. $\hat{S}^\pm(0) = \int_{-\infty}^{\infty} S^\pm(x)dx = \beta - \alpha \pm \frac{1}{\delta}$.
3. $\hat{S}^\pm(x) = 0$ *for* $|x| > \delta$.
4. *For any* $\epsilon > 0$, $S^\pm(z) \ll_{\delta,\alpha,\beta,\epsilon} \min(1, 1/|z|^2)$ *for* $\operatorname{Im}(z) \le \epsilon$.
5. $\hat{S}^\pm(z) = \frac{\sin \pi(\beta-\alpha)z}{\pi z} + O\left(\frac{1}{\delta}\right)$ *for* $|z| \le \delta$.

Proof. This is a specialization of [GG07, Lemma 2]. For a more detailed proof, see the survey articles of Montgomery [Mon78], Selberg [Sel91] and Vaaler [Vaa85]. □

3.4 Feasible pairs

From now on, we may assume that $\alpha = -\beta$ for simplicity, so that the Selberg minorant function depends on the two parameters β and δ. Therefore, we will write $S^-_{-\beta,\beta;\delta}(z) = S^-_{\beta;\delta}(z)$.

Definition 3.4.1 (Feasible pair)**.** *We say that (β, δ) is a feasible pair for $S^{\pm}_{\beta,\delta}(z)$ if*

$$\ell(\mu, S^{\pm}_{\beta;\delta}) := \mathrm{Re}\left(\int_{-\infty}^{\infty} \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{it}{2} + \mu\right) S^{\pm}_{\beta;\delta}(t)dt\right) - \hat{S}^{\pm}_{\beta;\delta}(0)\log\pi > 0,$$

for every $\mu \in \mathbb{C}$ with $\mathrm{Re}(\mu) \geq 0$. We also define the function

$$\begin{aligned}\eta^{\pm}(\beta, \delta) &= \inf_{\substack{\mu\in\mathbb{C}\\ \mathrm{Re}(\mu)\geq 0}} \left[\mathrm{Re}\left(\int_{-\infty}^{\infty} \frac{\Gamma'}{\Gamma}\left(\frac{1}{4} + \frac{it}{2} + \mu\right) S^{\pm}_{\beta;\delta}(t)dt\right) - \hat{S}^{\pm}_{\beta;\delta}(0)\log\pi\right] \\ &= \inf_{\substack{\mu\in\mathbb{C}\\ \mathrm{Re}(\mu)\geq 0}} \ell(\mu, S^{\pm}_{\beta;\delta}), \qquad (3.9)\end{aligned}$$

so that $\eta^{\pm}(\beta, \delta) > 0$ if and only if (β, δ) is a feasible pair.

Moreover, if (β, δ) is a feasible pair, we call $S^{\pm}_{\beta;\delta}(x)$ a feasible Selberg minorant/majorant function. Figure 3.2 shows the two dimensional surface $\ell(\mu, S^{-}_{\beta;\delta})$ for the variables $(\mathrm{Re}(\mu), \mathrm{Im}(\mu))$.

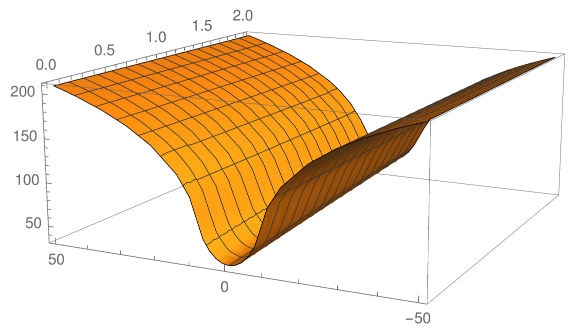

FIGURE 3.2: 3D Plot of $\ell(\mu, S^{-}_{\beta;\delta})$ for $(\mathrm{Re}(\mu), \mathrm{Im}(\mu))$, where $\mathrm{Re}(\mu) \geq 0$ for (β, δ) a feasible pair.

Definition 3.4.2 (Feasible set)**.** *The set*

$$\begin{aligned}G^{\pm} &= \left\{(\beta, \delta) \in (0, +\infty) \times \left(\frac{\log 2}{2\pi}, +\infty\right) \;:\; (\beta, \delta) \textit{ is a feasible pair}\right\} \\ &= \left\{(\beta, \delta) \in (0, +\infty) \times \left(\frac{\log 2}{2\pi}, +\infty\right) \;:\; \eta^{\pm}(\beta, \delta) > 0\right\} \qquad (3.10)\end{aligned}$$

is called a feasible set.

We define for each integer $m > 1$ the feasible region up to $\delta \leq \frac{\log m}{2\pi}$,

$$G^{\pm}_m = \left\{(\beta, \delta) \in G^{\pm} \;:\; \delta \leq \frac{\log m}{2\pi}\right\}. \qquad (3.11)$$

3.4.1 Linear combinations of Selberg minorant functions

Certain linear combinations of Selberg's functions may be used to improve the upper bound. The idea is to use a specific linear combination that makes the Fourier transform disappear at each, or certain points, $\frac{\log m}{2\pi}$ where the Weil explicit formula (3.5) contributes in the last sum. The following well-known fact will be used later.

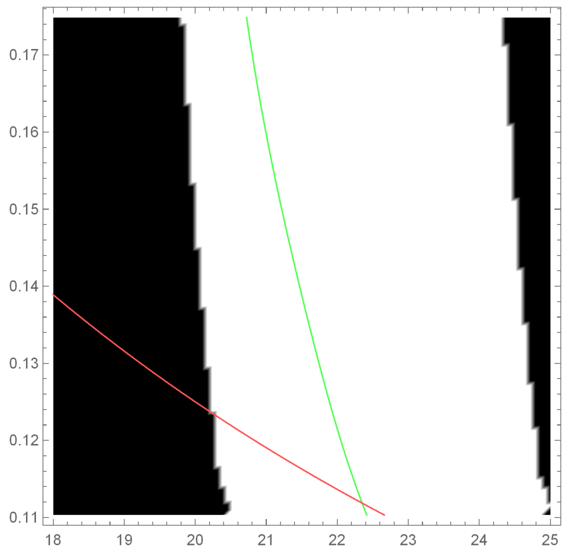

FIGURE 3.3: The region (β, δ) for $18 \leq \beta \leq 25$ and $\frac{\log 2}{2\pi} \leq \delta \leq \frac{\log 3}{2\pi}$. The feasible region lies on the right of the green line. The red line shows the hyperbola $\beta\delta = 2.5$, where the lowest point $\delta = \frac{\log 2}{2\pi}$, $\beta \approx 22.6$ is the result obtained in [BCF+15]. The white region is where $\hat{S}^{-}_{\beta,\delta}\left(\frac{\log 2}{2\pi}\right) > 0$ and the black region where it is negative.

Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ be n vectors in $\mathbb{R}^k$, and consider the convex hull H containing all these points. Then there exists a linear combination with non-negative coefficients which gives the zero vector if and only if $\boldsymbol{0}$ is contained in H.

We now state an abstract theorem for finding a possible upper bound using linear combinations of Selberg's minorant functions. The idea of the theorem will be used later in the chapter for a straightforward linear combination of two terms (this is due to an exponential increase of computational effort when we increase the terms to three or more).

Theorem 3.4.1. *Assume GRH, and denote by k_m the number of prime powers striclty smaller than m. Consider the map*

$$\psi_m : G^{-}_m \to \mathbb{R}^{k_m}$$
$$\psi_m\left((\beta, \delta)\right) := \left(\hat{S}^{-}_{\beta;\delta}\left(\frac{\log 2}{2\pi}\right), \ldots, \hat{S}^{-}_{\beta;\delta}\left(\frac{\log p^{k'}}{2\pi}\right)\right),$$

where $p^{k'}$ is the highest prime power less than m, to be the vector with components given by the values of the Fourier transform of $S^{-}_{\beta;\delta}(t)$ at $\frac{\log 2}{2\pi}$, $\frac{\log 3}{2\pi}$, ... and so on for all the prime powers less than m.

For each positive real number B, let

$$G^{-}_{m,B} = \{(\beta, \delta) \in G^{-}_m \ : \ \beta \leq B\}$$

and let $X_{m,B} = \psi_m(G^{-}_{m,B}) \subset \mathbb{R}^{k_m}$ be the image of $G_{m,B}$ through the map ψ_m.

Therefore, if we let

$$\beta_\infty = \inf_m \inf\{B > 0 \ : \ \boldsymbol{0} \in \text{hull}(X_{m,B})\},$$

then the gap between any two consecutive zeros is bounded by $2\beta_\infty$, i.e. $\mathrm{Gap}_* \leq 2\beta_\infty$.

The above theorem states that in certain cases we can find a linear combination of Selberg's functions that makes each Fourier transform of $S^-_{\beta;\delta}(t)$ disappear at each $\frac{\log m}{2\pi}$ if m is a prime power. In fact, if we keep m constant and increase B, then the set $X_{m,B}$ increases and so does its convex hull, which therefore has a better chance of containing the origin of $\mathbb{R}^{k_m}$.

If such B exists, then any larger B will automatically have the same desired property. In that case there will exist a smallest B with that property (that is, the convex hull of $X_{m,B}$ contains the origin of $\mathbb{R}^{k_m}$). If we denote this smallest B (which depends on m only) by β_m

$$\beta_m = \inf\{B > 0 \ : \ \mathbf{0} \in \mathrm{hull}(X_{m,B})\},$$

then its lowest value over all m must satisfy the requirements as well. The proof reads as follows.

Proof. If β_∞ is attained in $\{\beta_j \ : \ j > 1\}$, then consider that value of m such that β_m is the minimum. Otherwise, for any $\varepsilon > 0$ there exists an m such that $\beta_\infty + \varepsilon = \beta_m$. Fix $\varepsilon > 0$ and take the corresponding m. Consider the set $X_{m,\beta_m} \subset \mathbb{R}^{k_m}$, then by the above construction there are vectors $\mathbf{x}_1, \ldots, \mathbf{x}_{k_m} \in X_{m,\beta_m}$ such that their convex hull contains $\mathbf{0}$. By the fact of previous page regarding the convex hull, there are then non-negative constants $c_1, \ldots, c_{k_m}$ such that

$$c_1\mathbf{x}_1 + \ldots + c_{k_m}\mathbf{x}_{k_m} = \mathbf{0}. \tag{3.12}$$

Consider now the function

$$S(t) := \left(\sum_{j=1}^{k_m} c_j\right)^{-1} \sum_{j=1}^{k_m} c_j S^-_{\psi_m^{-1}(\mathbf{x}_j)}(t).$$

As ψ_m may not be injective, we will take the element in the set $\psi_m^{-1}(\mathbf{x}_j)$ such that β is minimal. Then applying Weil' explicit formula to $S(t)$ we obtain

$$\begin{aligned}\#\{\text{zeros in } (-\beta_m, \beta_m)\} \geq \sum_\gamma S(\gamma) &= \frac{\hat{S}(0)}{\pi}\log Q + \frac{1}{2\pi}\sum_{j=1}^{d} \ell(\mu_j, S) \\ &\geq \frac{\hat{S}(0)}{\pi}\log Q + \frac{d}{2\pi}\left(\sum_{j=1}^{k_m} c_j\right)^{-1} \sum_{j=1}^{k_m} c_j \eta(\psi_m^{-1}(\mathbf{x}_j))\end{aligned} \tag{3.13}$$

because (3.12) makes all the terms which are not prime powers vanish. Now, all the terms on the RHS of (3.13) are positive because of the construction of $S(t)$. Since this is true for any $\varepsilon > 0$ such that $\beta_m = \beta_\infty + \varepsilon$, it must be true for β_∞ as well. Now if $t_0 \in \mathbb{R}$, then any $S^-_{-\beta_\infty+t_0,\beta_\infty+t_0;\delta}(z)$ is a minorant of the characteristic function in the range $(-\beta_\infty + t_0, \beta_\infty + t_0)$. Its Fourier transform at 0 is given by Lemma 3.3.1:

$$\hat{S}^-_{-\beta_\infty+t_0,\beta_\infty+t_0;\delta}(0) = 2\beta_\infty - \frac{1}{\delta},$$

since the interval difference $\beta - \alpha$ remains constant. For the same β_m chosen at the beginning and for any $t_0 \neq 0$ the feasible set is such that

$$G_m^{-,t_0} := \left\{ (\beta,\delta) \in (0,+\infty) \times \left(\frac{\log 2}{2\pi}, \frac{\log m}{2\pi} \right) \; : \; \inf_{\substack{\mu \in \mathbb{C} \\ \mathrm{Re}(\mu) \geq 0}} \ell(\mu, S^-_{-\beta_m + t_0, \beta_m + t_0 \delta}) > 0 \right\}$$
$$\supset G_m^-.$$

Thus, $\psi_m(G_m^{-,t_0}) := X^{t_0}_{m,\beta_m} \supset X_{m,\beta_m}$ has a convex hull that contains $\mathbf{0}$. The linear combination of vectors can be thus be found as well, and hence the RHS of (3.13) must be positive. $\square$

3.4.2 Bounding the Dirichlet coefficients

The key idea in this note is to use the bounds on the coefficients (Ramanujan hypothesis and Luo-Rudnick-Sarnak bound) to obtain improved bounds. The first theorem illustrates the idea under the assumption of the Ramanujan hypothesis.

Theorem 3.4.2. *Assume the Ramanujan Hypothesis and GRH. Let $(\beta, \delta) \in G_M^-$ be a feasible pair for $S^-_{\beta,\delta}(t)$ such that*

$$\delta \in \left(\frac{\log(M-1)}{2\pi}, \frac{\log M}{2\pi} \right]$$

for a given $M \geq 2$, and suppose that

$$\eta(\beta,\delta) - \sum_p \sum_{\substack{k \\ p^k < M}} \frac{2\log p}{\sqrt{p^k}} \left| \hat{S}^-_{\beta;\delta}\left(\frac{k \log p}{2\pi} \right) \right| > 0. \tag{3.14}$$

Then every entire L-function has a nontrivial zero in every vertical interval of length 2β.

Proof. By the construction of (β,δ), we have $\eta(\beta,\delta) > 0$, and by the Weil explicit formula, the RHS of (3.5) equals

$$\Psi(Q,\mu_1,\ldots,\mu_d) := \frac{\hat{S}^-_{\beta,\delta}(0)}{\pi} \log Q + \frac{1}{2\pi} \sum_{j=1}^d \ell(\mu_j, S^-_{\beta;\delta})$$
$$- \frac{1}{\pi} \sum_p \sum_{\substack{k \\ p^k < M}} \left(\frac{\mathrm{Re}(\Lambda_L(p^k))}{\sqrt{p^k}} \hat{S}^-_{\beta;\delta}\left(\frac{k \log p}{2\pi} \right) \right).$$

As long as this term is positive for some (β,δ), then the statement of the theorem holds. Since this is true for any $Q \geq 1$ and any spectral parameters μ_j, then it must be positive for its infimum,

$$\inf_{\substack{Q \geq 1, \mu_j \in \mathbb{C} \\ \mathrm{Re}(\mu_j) \geq 0}} \Psi(Q,\mu_1,\ldots,\mu_d) \geq \frac{d}{2\pi} \eta(\beta,\delta)$$
$$- \frac{1}{2\pi} \sum_p \sum_{\substack{k \\ p^k < M}} \left(\frac{2\,\mathrm{Re}\left(\sum_{j=1}^d \alpha^k_{j,p} \right) \log p}{\sqrt{p^k}} \hat{S}^-_{\beta;\delta}\left(\frac{k \log p}{2\pi} \right) \right).$$

But for that to be positive, because of Corollary 3.3, we need

$$\sum_{p} \sum_{\substack{k \\ p^k < M}} \frac{2 \log p}{\sqrt{p^k}} \left| \hat{S}^-_{\beta;\delta} \left(\frac{k \log p}{2\pi} \right) \right| < \eta(\beta, \delta),$$

which is exactly equation (3.14). □

The next theorem illustrates a sufficient condition to be satisfied in the case where the Ramanujan hypothesis is not assumed, i.e. in the scenario where the unconditional result due to Luo-Rudnick-Sarnak [LRS99] (see Lemma 1.4.2) is assumed.

Theorem 3.4.3. *Assume GRH. Let $(\beta, \delta) \in G^-_M$ be a feasible pair for $S^-_{\beta;\delta}(t)$ such that*

$$\delta \in \left(\frac{\log(M-1)}{2\pi}, \frac{\log M}{2\pi} \right]$$

for a given $M \geq 2$, and suppose that

$$\eta(\beta, \delta) - \sum_{p} \sum_{\substack{k \\ p^k < M}} 2 \log p \left| \hat{S}^-_{\beta;\delta} \left(\frac{k \log p}{2\pi} \right) \right| > 0. \tag{3.15}$$

Then every entire L-function has a nontrivial zero in every vertical interval of length 2β.

Proof. Proceed as in the previous proof but here the bound is unconditional and thus we have

$$\left| \frac{\log p \operatorname{Re} \left(\sum_{j=1}^{d} \alpha^k_{j,p} \right)}{p^{k/2}} \right| \leq \frac{d p^{k/2} \log p}{p^{k/2}} = d \log p$$

because of Corollary 3.3. □

In our next theorem, the goal is to combine the theoretical idea put forward in Theorem 3.4.1 with the result of Theorem 3.4.2 for a convex combination of two Selberg minortant functions.

Theorem 3.4.4. *Assume GRH. Let $(\beta_1, \delta_1), (\beta_2, \delta_2) \in G^-_M$ be two feasible pairs such that $\delta_1, \delta_2 \in \left(\frac{\log(M-1)}{2\pi}, \frac{\log M}{2\pi} \right]$ for a given $M \geq 2$. Suppose that for an $m \geq 2$ which is a prime power the following inequalities hold:*

$$\hat{S}^-_{\beta_1;\delta_1} \left(\frac{\log m}{2\pi} \right) > 0 \text{ and } \hat{S}^-_{\beta_2;\delta_2} \left(\frac{\log m}{2\pi} \right) < 0,$$

or

$$\hat{S}^-_{\beta_1;\delta_1} \left(\frac{\log m}{2\pi} \right) < 0 \text{ and } \hat{S}^-_{\beta_2;\delta_2} \left(\frac{\log m}{2\pi} \right) > 0,$$

and set them to be c_1 and $-c_2$ such that either $c_1, c_2 > 0$ or $c_1, c_2 < 0$. Define the new function

$$S(t) := \frac{1}{c_1 + c_2} c_2 S^-_{\beta_1;\delta_1}(t) + c_1 S^-_{\beta_2;\delta_2}(t)$$

so that $\hat{S}(\frac{\log m}{2\pi}) = 0$. *If*

$$\frac{1}{c_1+c_2}(c_2\eta^-(\beta_1,\delta_1)+c_1\eta^-(\beta_2,\delta_2)) - \sum_p \sum_{\substack{k \\ p^k<M, p^k\neq m}} \frac{2\log p}{\sqrt{p^k}}\left|\hat{S}\left(\frac{k\log p}{2\pi}\right)\right| > 0 \quad (3.16)$$

on the Ramanujan hypothesis, or

$$\frac{1}{c_1+c_2}(c_2\eta^-(\beta_1,\delta_1)+c_1\eta^-(\beta_2,\delta_2)) - \sum_p \sum_{\substack{k \\ p^k<M, p^k\neq m}} 2\log p\left|\hat{S}\left(\frac{k\log p}{2\pi}\right)\right| > 0 \quad (3.17)$$

on the Luo-Rudnick-Sarnak bound. Then every entire L-function has a nontrivial zero in every vertical interval of length $2\max\{\beta_1,\beta_2\}$.

Proof. In this case, by the linearity of $\ell(\mu_j, f)$ in f, the infimum of the RHS of (3.5) over $Q \geq 1$ and $\mathrm{Re}(\mu_j) \geq 0$ is greater or equal than

$$\frac{d}{2\pi}\frac{1}{c_1+c_2}(c_2\eta^-(\beta_1,\delta_1)+c_1\eta^-(\beta_2,\delta_2))$$
$$-\frac{d}{2\pi}\sum_p \sum_{\substack{k \\ p^k<M}} \left(\frac{2\,\mathrm{Re}\left(\sum_{j=1}^d \alpha_{j,p}^k\right)\log p}{\sqrt{p^k}}\hat{S}\left(\frac{k\log p}{2\pi}\right)\right)$$

which is positive whenever (3.16) holds, and the term in the sum corresponding to $\hat{S}(\frac{\log m}{2\pi})$ disappears because of the construction of S. □

3.4.3 Proof of Theorem 3.1.1

We are now ready to prove our main result, namely Theorem 3.1.1. As mentioned earlier, the key to obtaining the two bounds is to use a convex linear combination of Selberg's minorant functions as in Theorem 3.4.4 that makes its Fourier transform disappear at the point $\frac{\log 4}{2\pi}$ in the case of the Ramanujan hypothesis, and at the point $\frac{\log 3}{2\pi}$ in the case of the Luo-Rudnick-Sarnak bound.

Proof of Theorem 3.1.1. On the Ramanujan hypothesis.
For $M = 7$ and the pairs

$$(\beta_1,\delta_1) = \left(18.2, \frac{\log 7}{2\pi}\right) \quad \text{as well as} \quad (\beta_2,\delta_2) = \left(20.770, \frac{\log 7}{2\pi}\right),$$

and $m = 4$ one can show that the conditions of Theorem 3.4.4 are satisfied and

$$\frac{1}{c_1+c_2}(c_2\eta^-(\beta_1,\delta_1)+c_1\eta^-(\beta_2,\delta_2))$$
$$-\frac{2\log 2}{\sqrt{2}}\left|\hat{S}\left(\frac{\log 2}{2\pi}\right)\right| - \frac{2\log 3}{\sqrt{3}}\left|\hat{S}\left(\frac{\log 3}{2\pi}\right)\right| - \frac{2\log 5}{\sqrt{5}}\left|\hat{S}\left(\frac{\log 5}{2\pi}\right)\right| \quad (3.18)$$

is positive, as shown in Figure 3.4. Specifically, for $\beta_2 = 20.770$ it gives the value of 0.000825207.

Theorem 3.4.4 also holds for $\max\{\beta_1,\beta_2\} = \beta_2 = 20.770$.

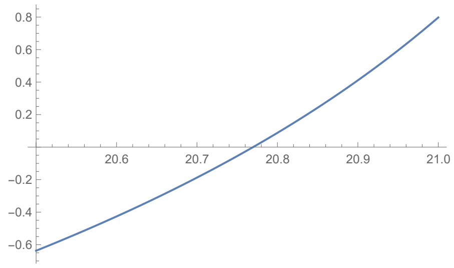

FIGURE 3.4: Plot of the function (3.18) for $\beta_1 = 18.2$ and $\beta_2 \in (20.5, 21.0)$ assuming the Ramanujan hypothesis.

On the Luo-Rudnick-Sarnak bound.
For $M = 7$ we use again Theorem 3.4.4 applied to the two pairs

$$(\beta_1, \delta_1) = \left(19.7, \frac{\log 7}{2\pi}\right) \quad \text{as well as} \quad (\beta_2, \delta_2) = \left(21.705., \frac{\log 7}{2\pi}\right),$$

and with $m = 3$. The given choice satisfies the requirements of Theorem 3.4.4 and the condition (3.17) reduces to

$$\frac{1}{c_1 + c_2}(c_1\eta^-(\beta_1, \delta_1) + c_2\eta^-(\beta_2, \delta_2)) \\ - 2\log 2 \left|\hat{S}\left(\frac{\log 2}{2\pi}\right)\right| - 2\log 2 \left|\hat{S}\left(\frac{\log 4}{2\pi}\right)\right| - 2\log 5 \left|\hat{S}\left(\frac{\log 5}{2\pi}\right)\right|, \tag{3.19}$$

which is positive for the given pair for $\beta_2 \geq 21.705$ as shown in Figure 3.5. Specifically, for $\beta_2 = 21.705$ it gives the value of 0.00655263. This ends the proof. □

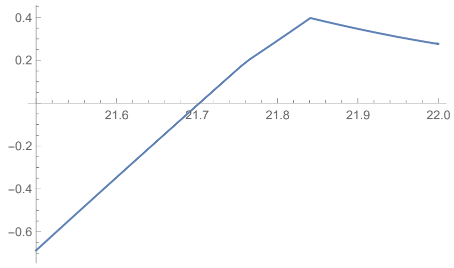

FIGURE 3.5: Plot of the function (3.19) for $\beta_1 = 19.7$ and $\beta_2 \in (21.5, 22)$ on the Luo-Rudnick-Sarnak bound.

3.4.4 The lowest bound

The example of L-function shown in [BCF+15, §3] shows that there are L-functions whose largest gaps between zeros are greater than the one of the Riemann zeta-function. This is the case where the L-function has non-real spectral parameters, because Miller [Mil02] proved that the largest gap between zeros of the Riemann zeta-function is greater than any other among L-functions of real spectral parameters.

Therefore, there must be a general lowest upper bound which is higher than the largest gap between zeros of the Riemann zeta-function. One could try to guess what this lowest bound might be.

Open Problem 3.4.1. *Is it true that at least a zero γ must be contained in $[t_0, t_0 + 2\beta_-]$ for any $t_0 \in \mathbb{R}$, where*

$$\beta_- := \inf\{\beta > 0 \ : \ \eta(\beta, \delta) = 0\} \approx 17.845?$$

In [BCF+15], a threshold of 36 was suggested. The slightly lower value of $2\beta_- \approx 35.69$ might be reasonable since it is the lowest possible β of the feasible region where the positivity of the Γ-terms is guaranteed. We also raise the following problem.

Open Problem 3.4.2. *Is the value $2\beta_-$ an optimal upper bound for the largest gap between zeros among entire L-functions, i.e. is $\mathrm{Gap}_* = 2\beta_-$? In other words, is it true that for any $\varepsilon > 0$ there is an entire L-function $L(s)$ whose largest gap Gap_L is either*

$$\mathrm{Gap}_L = 2\beta_-,$$

or

$$\mathrm{Gap}_L \geq 2\beta_- - \varepsilon \,?$$

The first case above might be too optimistic, the second option might be more probable. In fact, the lowest upper bound could be an accumulation point of zeros of certain L-functions with non-real spectral parameters.

3.5 Nonexistence of certain entire L-functions

There are other interesting applications of the Weil explicit formula. For instance, by using the Selberg majorant function for a $\delta \in \left(\frac{\log 2}{2\pi}, \frac{\log 3}{2\pi}\right]$, we can bound the real part of the first Dirichlet coefficient $b(2)$ in the following way.

Theorem 3.5.1. *Assume GRH. Let $\beta \geq 0$ and suppose that $\delta \in \left(\frac{\log 2}{2\pi}, \frac{\log 3}{2\pi}\right]$. If $\hat{S}^+_{\beta;\delta}\left(\frac{\log 2}{2\pi}\right) > 0$ for some δ in that interval, then*

$$\mathrm{Re}(b(2)) \leq \frac{\sqrt{2}}{\log 2}\left(\hat{S}^+_{\beta;\delta}(0)\log Q + \frac{1}{2}\sum_{j=1}^{d}\ell(\mu_j, S^+_{\beta,\delta})\right)\hat{S}^+_{\beta;\delta}\left(\frac{\log 2}{2\pi}\right)^{-1}.$$

Proof. Take the Selberg majorant function $S^+_{\beta;\delta}(z)$, for $\delta \in \left(\frac{\log 2}{2\pi}, \frac{\log 3}{2\pi}\right]$. Thus, since $S^+_{\beta,\delta}(x) \geq 0$ for all real x,

$$0 \leq \#\{\text{ zeros in } (-\beta,\beta)\} \leq \sum_{\gamma} S^+_{\beta;\delta}(\gamma),$$

which translates to

$$0 \leq \frac{\hat{S}^+_{\beta;\delta}(0)}{\pi}\log Q + \frac{1}{2\pi}\sum_{j=1}^{d}\ell(\mu_j, S^+_{\beta;\delta}) - \frac{1}{\pi}\left(\frac{\mathrm{Re}(b(2)\log 2)}{\sqrt{2}}\hat{S}^+_{\beta;\delta}\left(\frac{\log 2}{2\pi}\right)\right).$$

That is,

$$\frac{\mathrm{Re}(b(2)\log 2)}{\sqrt{2}}\hat{S}^+_{\beta;\delta}\left(\frac{\log 2}{2\pi}\right) \leq \hat{S}^+_{\beta;\delta}(0)\log Q + \frac{1}{2}\sum_{j=1}^{d}\ell(\mu_j, S^+_{\beta;\delta}).$$

Then, because $\hat{S}^{+}_{\beta;\delta}\left(\frac{\log 2}{2\pi}\right) > 0$,

$$\operatorname{Re}(b(2)) \leq \frac{\sqrt{2}}{\log 2}\left(\hat{S}^{+}_{\beta;\delta}(0)\log Q + \frac{1}{2}\sum_{j=1}^{d}\ell(\mu_j, S^{+}_{\beta;\delta})\right)\hat{S}^{+}_{\beta;\delta}\left(\frac{\log 2}{2\pi}\right)^{-1},$$

and the proof is now finished. $\square$

Under the Ramanujan hypothesis, we can use this last result to prove that certain entire L-functions having specific functional equations cannot exist.

Corollary 3.5.1. *Assume GRH and the Ramanujan hypothesis. If, for some (β, δ) as in the Theorem 3.5.1, the condition*

$$\frac{\sqrt{2}}{\log 2}\left(\hat{S}^{+}_{\beta;\delta}(0)\log Q + \frac{1}{2}\sum_{j=1}^{d}\ell(\mu_j, S^{+}_{\beta;\delta})\right)\hat{S}^{+}_{\beta;\delta}\left(\frac{\log 2}{2\pi}\right)^{-1} < -d$$

holds for some spectral parameter choice, then there is no entire L-function having these spectral parameters.

Corollary 3.5.2. *Assume GRH and the Ramanujan hypothesis and let (β, δ) be such that the inequality of Corollary 3.5.1 is satisfied. There is no entire L-function such that it has d copies of $\Gamma_{\mathbb{R}}(s)$ terms in the functional equation if*

$$Q < \exp\left(\left(-\frac{d\log 2}{\sqrt{2}}\hat{S}^{+}_{\beta;\delta}\left(\frac{\log 2}{2\pi}\right) - \frac{d}{2}\ell(0, S^{+}_{\beta;\delta})\right)\left(2\beta + \frac{1}{\delta}\right)^{-1}\right). \tag{3.20}$$

In particular, for L-functions with degree up to 5, no entire L-function with functional equation of the form

$$\Lambda(s) = Q^{s}\Gamma_{\mathbb{R}}(s)^{d}L(s) = \varepsilon\bar{\Lambda}(1-s)$$

exists for all values of Q and N less than the values in the following table.

d	2	3	4	5
Q	< 4.24243	< 8.73821	< 17.9982	< 37.0713
N	< 17	< 77	< 324	< 1375

Proof. Apply Corollary 3.5.1 to the specific case,

$$\left(2\beta + \frac{1}{\delta}\right)\log Q + \frac{d}{2}\ell(0, S^{+}_{\beta;\delta}) < -\frac{d\log 2}{\sqrt{2}}\hat{S}^{+}_{\beta;\delta}\left(\frac{\log 2}{2\pi}\right)$$

and solve for Q. The particular cases follow from evaluating (3.20) with the parameters $(\beta, \delta) = (0.5, \frac{\log 3}{2\pi})$ using Mathematica. $\square$

For the particular point $(\beta, \delta) = (0.5, \frac{\log 3}{2\pi})$, the term $\ell(0, S^{+}_{\beta;\delta})$ is negative and this shows the exponential increase which depends on the degree of the upper bound.

All computations were made using Mathematica 10.3.

Chapter 4

Explicit formulas of a generalized Ramanujan sum

4.1 Introduction

This chapter is taken from [KR16], published in the International Journal of Number Theory, in collaboration with N. Robles.

In [Ram18] Ramanujan introduced the following trigonometric sum.

Definition 4.1.1. *The Ramanujan sum is defined by*

$$c_q(n) = \sum_{(h,q)=1} e^{2\pi i n h/q}, \tag{4.1}$$

where q and n are in $\mathbb{N}$ and the summation is over a reduced residue system $\bmod\ q$.

Many properties were derived in [Ram18] and elaborated in [Har21]. Cohen [Coh49] generalized this arithmetical function in the following way.

Definition 4.1.2. *Let $\beta \in \mathbb{N}$. The $c_q^{(\beta)}(n)$ sum is defined by*

$$c_q^{(\beta)}(n) = \sum_{(h,q^\beta)_\beta=1} e^{2\pi i n h/q^\beta}, \tag{4.2}$$

where h ranges over the non-negative integers less than q^β such that h and q^β have no common β-th power divisors other than 1.

It follows immediately that when $\beta = 1$, (4.2) becomes the Ramanujan sum (4.1). Among the most important properties of $c_q^{(\beta)}(n)$ we mention that it is a multiplicative function of q, i.e.

$$c_{pq}^{(\beta)}(n) = c_p^{(\beta)}(n) c_q^{(\beta)}(n), \quad (p,q) = 1.$$

The purpose of this chapter is to derive explicit formulas involving $c_q^{(\beta)}(n)$ in terms of the non-trivial zeros ρ of the Riemann zeta-function and establish arithmetic theorems.

Definition 4.1.3. *Let $z \in \mathbb{C}$. The generalized divisor function $\sigma_z^{(\beta)}(n)$ is the sum of the z^{th} powers of those divisors of n which are β^{th} powers of integers, i.e.*

$$\sigma_z^{(\beta)}(n) = \sum_{d^\beta | n} d^{\beta z}.$$

The object of study is the following.

Definition 4.1.4. *For $x \geq 1$, we define*

$$\mathfrak{C}^{(\beta)}(n,x) = \sum_{q \leqslant x} c_q^{(\beta)}(n).$$

For technical reasons we set

$$\mathfrak{C}^{\sharp,(\beta)}(n,x) = \begin{cases} \mathfrak{C}^{(\beta)}(n,x), & \text{if } x \notin \mathbb{N}, \\ \mathfrak{C}^{(\beta)}(n,x) - \frac{1}{2}c_x^{(\beta)}(n), & \text{if } x \in \mathbb{N}. \end{cases} \tag{4.3}$$

The explicit formula for $\mathfrak{C}^{\sharp,(\beta)}(n,x)$ is then as follows.

Theorem 4.1.1. *Let ρ and ρ_m denote non-trivial zeros of $\zeta(s)$ of multiplicity 1 and $m \geq 2$ respectively. Fix integers β, n. There is an $1 > \varepsilon > 0$ and a $T_0 = T_0(\varepsilon)$ such that (4.11) and (4.12) hold for a sequence T_ν and*

$$\mathfrak{C}^{\sharp,(\beta)}(n,x) = -2\sigma_1^{(\beta)}(n) + \sum_{|\gamma|<T_\nu} \frac{\sigma_{1-\rho/\beta}^{(\beta)}(n)}{\zeta'(\rho)} \frac{x^\rho}{\rho} + K_{T_\nu}(x) - \sum_{k=1}^{\infty} \frac{(-1)^k (2\pi/x)^{2k}}{(2k)! k \zeta(2k+1)} \sigma_{1+2k/\beta}^{(\beta)}(n) + E_{T_\nu}(x),$$

where the error term satisfies

$$E_{T_\nu}(x) \ll \frac{x \log x}{T_\nu^{1-\varepsilon}},$$

and where for the zeros of multiplicity $m \geq 2$ we have

$$K_{T_\nu}(x) = \sum_{m \geqslant 2} \sum_{|\gamma_m|<T_\nu} \kappa(\rho_m, x),$$

$$\kappa(\rho_m, x) = \frac{1}{(m-1)!} \lim_{s \to \rho_m} \frac{d^{m-1}}{ds^{m-1}} \left((s-\rho_m)^m \frac{\sigma_{1-s/\beta}^{(\beta)}(n)}{\zeta(s)} \frac{x^s}{s} \right).$$

Moreover, in the limit $\nu \to \infty$ we have

$$\mathfrak{C}^{\sharp,(\beta)}(n,x) = -2\sigma_1^{(\beta)}(n) + \lim_{\nu \to \infty} \sum_{|\gamma|<T_\nu} \frac{\sigma_{1-\rho/\beta}^{(\beta)}(n)}{\zeta'(\rho)} \frac{x^\rho}{\rho} + \lim_{\nu \to \infty} K_{T_\nu}(x) - \sum_{k=1}^{\infty} \frac{(-1)^k (2\pi/x)^{2k}}{(2k)! k \zeta(2k+1)} \sigma_{1+2k/\beta}^{(\beta)}(n).$$

More information about the existence and the construction of the sequence T_ν is given in the next section.

The moments of $\mathfrak{C}^{\sharp,(\beta)}(n,x)$ were studied by Robles and Roy [RR16], where they proved estimates for the first and second moments.

The next result is a generalization of a well-known theorem of Ramanujan which is of the same depth as the prime number theorem.

Theorem 4.1.2. *For fixed β and n in $\mathbb{N}$, we have*

$$\frac{\sigma^{(\beta)}_{1-s/\beta}(n)}{\zeta(s)} = \sum_{q=1}^{\infty} \frac{c_q^{(\beta)}(n)}{q^s} \tag{4.4}$$

at all points on the line $\mathrm{Re}(s) = 1$.

Corollary 4.1.1. *Let $\beta \in \mathbb{N}$. Then*

$$\sum_{q=1}^{\infty} \frac{c_q^{(\beta)}(n)}{q} = 0, \quad \beta \geq 1, \quad \text{and} \quad \sum_{q=1}^{\infty} \frac{c_q^{(\beta)}(n)}{q^{\beta}} = \begin{cases} \frac{\sigma_0^{(\beta)}(n)}{\zeta(\beta)} & \text{if } \beta > 1, \\ 0 & \text{if } \beta = 1. \end{cases} \tag{4.5}$$

In particular

$$\sum_{q=1}^{\infty} \frac{c_q(n)}{q} = 0 \quad \text{and} \quad \sum_{q=1}^{\infty} \frac{\mu(q)}{q} = 0. \tag{4.6}$$

It is possible to further extend the validity of (4.5) deeper into the critical strip, however, this is done at the cost of the Riemann hypothesis.

Theorem 4.1.3. *Let $\beta, n \in \mathbb{N}$. The Riemann hypothesis is true if and only if*

$$\sum_{q=1}^{\infty} \frac{c_q^{(\beta)}(n)}{q^s} \tag{4.7}$$

is convergent and its sum is $\sigma^{(\beta)}_{1-s/\beta}(n)/\zeta(s)$, for every s with $\sigma > \frac{1}{2}$.

This is a generalization of a theorem proved by Littlewood (see [Lit12] and §14.25 of [Tit86]) for the special case where $n = 1$.

Theorem 4.1.4. *A necessary and sufficient condition for the Riemann hypothesis is*

$$\mathfrak{C}^{(\beta)}(n, x) \ll_{n,\beta} x^{\frac{1}{2}+\varepsilon} \tag{4.8}$$

for every $\varepsilon > 0$.

We recall that the von Mangolt function $\Lambda(n)$ may be defined by

$$\Lambda(n) = \sum_{d\delta=n} \mu(d) \log \delta.$$

Since $c_q^{(\beta)}(n)$ is a generalization of the Möbius function, we wish to construct a new $\Lambda(n)$ that incorporates the arithmetic information encoded in the variable q and the parameter β.

Definition 4.1.5. *For $\beta, k, m \in \mathbb{N}$ the generalized von Mangoldt function is defined as*

$$\Lambda^{(\beta)}_{k,m}(n) = \sum_{d\delta=n} c_d^{(\beta)}(m) \log^k \delta.$$

We note the special case $\Lambda_{1,1}^{(1)}(n) = \Lambda(n)$. We will, for the sake of simplicity, work with $k = 1$. The generalization for $k > 1$ requires dealing with results involving (computable) polynomials of degree $k - 1$, see for instance §12.4 of [Ivi85] as well as [Ivi75] and [Ivi77].

Definition 4.1.6. *The generalized Chebyshev function $\psi_m^{(\beta)}(x)$ and $\psi_m^{\sharp,(\beta)}(x)$ are defined by*

$$\psi_m^{(\beta)}(x) = \sum_{n \leqslant x} \Lambda_{1,m}^{(\beta)}(n), \quad \text{and} \quad \psi_m^{\sharp,(\beta)}(x) = \frac{1}{2}(\psi_m^{(\beta)}(x^+) + \psi_m^{(\beta)}(x^-)).$$

for $\beta, m \in \mathbb{N}$.

The explicit formula for the generalized Chebyshev function is given by the following result.

Theorem 4.1.5. *Let $c > 1$, $\beta \in \mathbb{N}$, $x > m$, $T \geq 2$ and let $\langle x \rangle_\beta$ denote the distance from x to the nearest interger n such that $\Lambda_{1,m}^{\beta}(n)$ is not zero (other than x itself). Then*

$$\begin{aligned}\psi_m^{\sharp,(\beta)}(x) = \sigma_{1-1/\beta}^{(\beta)}(m)x - \sum_{|\gamma| \leqslant T} \sigma_{1-\rho/\beta}^{(\beta)}(m)\frac{x^\rho}{\rho} \\ - \sigma_1^{(\beta)}(m)\log(2\pi) - \sum_{k=1}^{\infty} \sigma_{1+2k/\beta}^{(\beta)}(m)\frac{x^{-2k}}{2k} + R(x,T),\end{aligned}$$

where

$$R(x,T) \ll x^\varepsilon \min\left(1, \frac{x}{T\langle x \rangle_\beta}\right) + \frac{x^{1+\varepsilon}\log x}{T} + \frac{x\log^2 T}{T},$$

for all $\varepsilon > 0$.

Taking into account the standard zero-free region of the Riemann-zeta function we obtain

Theorem 4.1.6. *We have*

$$|\psi_m^{(\beta)}(x) - \sigma_{1-1/\beta}^{(\beta)}(m)x| \ll x^{1+\varepsilon}e^{-c_2(\log x)^{1/2}},$$

for $\beta \in \mathbb{N}$.

Moreover, on the Riemann hypothesis, one naturally obtains a better error term.

Theorem 4.1.7. *Assume RH. For $\beta \in \mathbb{N}$,*

$$\psi_m^{(\beta)}(x) = \sigma_{1-1/\beta}^{(\beta)}(m)x + O(x^{1/2+\varepsilon})$$

for each $\varepsilon > 0$.

Our next set of results is concerned with a generalization of a function introduced by Bartz [Bar91a; Bar91b]. The function introduced by Bartz was later used by Kaczorowski in [Kac07] to study sums involving the Möbius function twisted by the cosine function. Let us set $\mathbb{H} = \{x + iy,\ x \in \mathbb{R},\ y > 0\}$.

Definition 4.1.7. *Suppose that $z \in \mathbb{H}$, we define the ϖ function by*

$$\varpi_n^{(\beta)}(z) = \lim_{m\to\infty} \sum_{\substack{\rho \\ 0<\operatorname{Im}\rho<T_m}} \frac{\sigma_{1-\rho/\beta}^{(\beta)}(n)}{\zeta'(\rho)} e^{\rho z}. \tag{4.9}$$

The goal is to describe the analytic character of $\varpi_n^{(\beta)}(z)$. Specifically, we will construct its analytic continuation to a meromorphic function of z on the whole complex plane and prove that it satisfies a functional equation. This functional equation takes into account values of $\varpi_n^{(\beta)}(z)$ at z and at $\bar{z}$; therefore one may deduce the behavior of $\varpi_n^{(\beta)}(z)$ for $\mathrm{Im}(z) < 0$. Finally, we will study the singularities and residues of $\varpi_n^{(\beta)}(z)$.

Theorem 4.1.8. *The function $\varpi_n^{(\beta)}(z)$ is holomorphic on the upper half-plane $\mathbb{H}$, and for $z \in \mathbb{H}$ we have*

$$2\pi i \varpi_n^{(\beta)}(z) = \varpi_{1,n}^{(\beta)}(z) + \varpi_{2,n}^{(\beta)}(z) - e^{3z/2} \sum_{q=1}^{\infty} \frac{c_q^{(\beta)}(n)}{q^{3/2}(z - \log q)},$$

where the last term on the right hand-side is a meromorphic function on the whole complex plane with poles at $z = \log q$ whenever $c_q^{(\beta)}(n)$ is not equal to zero. Moreover,

$$\varpi_{1,n}^{(\beta)}(z) = \int_{-1/2+i\infty}^{-1/2} \frac{\sigma_{1-s/\beta}^{\beta}(n)}{\zeta(s)} e^{sz} ds$$

is analytic on $\mathbb{H}$ and

$$\varpi_{2,n}^{(\beta)}(z) = \int_{-1/2}^{3/2} \frac{\sigma_{1-s/\beta}^{\beta}(n)}{\zeta(s)} e^{sz} ds$$

is entire.

Remark 4.1.1. *This is done on the assumption that the non-trivial zeros are all simple. This is done for the sake of clarity, since straightforward modifications are needed to relax this assumption. See §4.7 for further details.*

Theorem 4.1.9. *The function $\varpi_n^{(\beta)}(z)$ can be continued analytically to a meromorphic function on $\mathbb{C}$ which satisfies the functional equation*

$$\begin{aligned} &\varpi_n^{(\beta)}(z) + \overline{\varpi_n^{(\beta)}(\bar{z})} \\ =&A_n^{(\beta)}(z) = -\sum_{q=1}^{\infty} \frac{\mu(q)}{q} \sum_{k=0}^{\infty} \frac{1}{k!} \left\{ \left(e^{-z} \frac{2\pi i}{q} \right)^k + \left(-e^{-z} \frac{2\pi i}{q} \right)^k \right\} \sigma_{1+k/\beta}^{(\beta)}(n), \end{aligned} \tag{4.10}$$

where the function $A_n^{(\beta)}(z)$ is entire and satisfies

$$A_n^{(\beta)}(z) = 2 \sum_{k=1}^{\infty} \frac{(-1)^k (2\pi)^{2k}}{(2k)!} \frac{e^{-2kz} \sigma_{1+k/\beta}^{(\beta)}(n)}{\zeta(1+2k)}.$$

Theorem 4.1.10. *The only singularities of $\varpi_n^{(\beta)}(z)$ are simple poles at the points $z = \log q$ on the real axis, where q is an integer such that $c_q^{(\beta)}(n) \neq 0$, with residue*

$$\operatorname*{res}_{z=\log n} \varpi_n^{(\beta)}(z) = -\frac{1}{2\pi i} c_q^{(\beta)}(n).$$

4.2 Proof of Theorem 4.1.1

In order to obtain unconditional results we use an idea put forward by Bartz [Bar91b]. The key is to use the following result of Montgomery, see [Mon77] and Theorem 9.4 of [Ivi85].

Lemma 4.2.1. *For any given $\varepsilon > 0$ there exists a real $T_0 = T_0(\varepsilon)$ such that for $T \geq T_0$ the following holds: between T and $2T$ there exists a value of t for which*

$$|\zeta(\sigma \pm it)|^{-1} < c_1 t^{\varepsilon} \text{ for } -1 \leq \sigma \leq 2,$$

with an absolute constant $c_1 > 0$ (not depending on ε).

That is, for each $\varepsilon > 0$, there is a sequence T_ν, where

$$2^{\nu-1}T_0(\varepsilon) \leq T_\nu \leq 2^{\nu}T_0(\varepsilon),\ \nu = 1, 2, 3, \ldots \tag{4.11}$$

such that

$$|\zeta(\sigma \pm iT_\nu)|^{-1} < c_1 T_\nu^{\varepsilon} \text{ for } -1 \leq \sigma \leq 2. \tag{4.12}$$

Finally, towards the end we will need the following bracketing condition: T_m ($m \leq T_m \leq m+1$) are chosen so that

$$|\zeta(\sigma + iT_m)|^{-1} < T_m^{c_2} \tag{4.13}$$

for $-1 \leq \sigma \leq 2$ and c_2 is an absolute constant. The existence of such a sequence of T_m is guaranteed by Theorem 9.7 of [Tit86], which itself is a result of Valiron, [Val14].
We will use either bracketing (4.11)-(4.12) or (4.13) depending on the necessity. These choices will lead to different bracketings of the sum over the zeros in the various explicit formulas appearing in the theorems of this note.

Next, we go back to the generalized divisor function $\sigma_z^{(\beta)}(n)$ with this first immediate result.

Lemma 4.2.2. *The generalized divisor function $\sigma_z^{(\beta)}(n)$ satisfies the following bound for $z \in \mathbb{C}$, $n \in \mathbb{N}$*

$$|\sigma_z^{(\beta)}(n)| \leq \sigma_{\mathrm{Re}(z)}^{(\beta)}(n) \leq n^{\beta \max(0, \mathrm{Re}(z))+1}.$$

In [Coh49] the following two properties of $c_q^{(\beta)}(n)$ are derived.

Lemma 4.2.3. *For β and n integers one has*

$$c_q^{(\beta)}(n) = \sum_{\substack{d|q \\ d^\beta|n}} \mu\left(\frac{q}{d}\right) d^\beta,$$

where μ denotes the Möbius function.

The relation between $c_q^{(\beta)}(n)$ and $\sigma_z^{(\beta)}(n)$ is given by the Dirichlet sum of $c_q^{(\beta)}(n)$ over q.

Lemma 4.2.4. *For $\mathrm{Re}(s) > 1$ and $\beta \in \mathbb{N}$ one has*

$$\sum_{q=1}^{\infty} \frac{c_q^{(\beta)}(n)}{q^{\beta s}} = \frac{\sigma_{1-s}^{(\beta)}(n)}{\zeta(\beta s)}.$$

We can now start the proof of Theorem 4.1.1. From Lemma 4.2.3 one has the following bound

$$|c_q^{(\beta)}(n)| \leqslant \sum_{\substack{d|q \\ d^\beta|n}} d^\beta \leqslant \sum_{d^\beta|n} d^\beta = \sigma_1^{(\beta)}(n).$$

Suppose x is a fixed non-integer. Let us now consider the positively oriented path $\mathcal{C}$ made up of the line segments $[c-iT, c+iT, -2N-1+iT, -2N-1-iT]$ where T is not the ordinate of a non-trivial zero. We set $a_q = c_q^{(\beta)}(n)$ and we use the lemma in §3.12 of [Tit86] to see that we can take $\psi(q) = \sigma_1^{(\beta)}(n)$. We note that for $\sigma > 1$ we have

$$\sum_{q=1}^{\infty} \frac{|c_q^{(\beta)}(n)|}{q^\sigma} \leqslant \sigma_1^{(\beta)}(n) \sum_{q=1}^{\infty} \frac{1}{q^\sigma} = \sigma_1^{(\beta)}(n)\zeta(\sigma) \ll \frac{1}{\sigma-1}$$

so that $\alpha = 1$. Moreover, if in that lemma we put $s = 0$, $c = 1 + 1/\log x$ and replace w by s, then we obtain

$$\mathfrak{C}_0^{(\beta)}(n,x) = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{\sigma_{1-s/\beta}^{(\beta)}(n)}{\zeta(s)} \frac{x^s}{s} ds + E_{1,T}(x),$$

where $E_{1,T}(x)$ is an error term that will be evaluated later. If x is an integer, then $\frac{1}{2}c_x^{(\beta)}(n)$ is to be subtracted from the left-hand side. Then, by residue calculus we have

$$\frac{1}{2\pi i} \oint_{\mathcal{C}} \frac{\sigma_{1-s/\beta}^{(\beta)}(n)}{\zeta(s)} \frac{x^s}{s} ds = R_0 + R_\rho(T) + K(x,T) + R_{-2k}(N),$$

where each term is given by the residues inside $\mathcal{C}$

$$R_0 = \operatorname*{Res}_{s=0} \frac{\sigma_{1-s/\beta}^{(\beta)}(n)}{\zeta(s)} \frac{x^s}{s} = -2\sigma_1^{(\beta)}(n),$$

and for $k = 1, 2, 3, \cdots$ we have

$$\begin{aligned} R_{-2k}(N) &= \sum_{k=1}^{N} \operatorname*{Res}_{s=-2k} \frac{\sigma_{1-s/\beta}^{(\beta)}(n)}{\zeta(s)} \frac{x^s}{s} = \sum_{k=1}^{N} \frac{\sigma_{1+2k/\beta}^{(\beta)}(n)}{\zeta'(-2k)} \frac{x^{-2k}}{-2k} \\ &= \sum_{k=1}^{N} \frac{(-1)^{k-1}(2\pi/x)^{2k}}{(2k)!k\zeta(2k+1)} \sigma_{1+2k/\beta}^{(\beta)}(n). \end{aligned}$$

For the non-trivial zeros we must distinguish two cases. For the simple zeros ρ we have

$$R_\rho(T) = \sum_{|\gamma|<T} \operatorname*{Res}_{s=\rho} \frac{\sigma_{1-s/\beta}^{(\beta)}(n)}{\zeta(s)} \frac{x^s}{s} = \sum_{|\gamma|<T} \frac{\sigma_{1-\rho/\beta}^{(\beta)}(n)}{\zeta'(\rho)} \frac{x^\rho}{\rho},$$

and by the formula for the residues of order m we see that $K(x,T)$ is of the form indicated in the statement of the theorem. We now bound the vertical integral on the far left

$$\begin{aligned} \int_{-2N-1-iT}^{-2N-1+iT} \frac{\sigma_{1-s/\beta}^{(\beta)}(n)}{\zeta(s)} \frac{x^s}{s} ds &= \int_{2N+2-iT}^{2N+2+iT} \frac{\sigma_{1-(1-s)/\beta}^{(\beta)}(n)}{\zeta(1-s)} \frac{x^{1-s}}{1-s} ds \\ &= \int_{2N+2-iT}^{2N+2+iT} \frac{x^{1-s}}{1-s} \frac{\sigma_{1-(1-s)/\beta}^{(\beta)}(n) 2^{s-1}\pi^s}{\cos(\frac{\pi s}{2})\Gamma(s)} \frac{1}{\zeta(s)} ds \end{aligned}$$

$$\ll \int_{-T}^{T} \frac{1}{T}\left(\frac{2\pi}{x}\right)^{2N+2} e^{(2N+3)\log(n)+2N+2-(2N+\frac{3}{2})\log(2N+2)}dt,$$

since by the use of Lemma 4.2.2 we have $\sigma^{(\beta)}_{1-(1-s)/\beta}(n) \ll \sigma^{(\beta)}_{1-(1-2-2N)/\beta}(n) \ll n^{2N+3}$. This tends to zero as $N \to \infty$, for a fixed T and a fixed n. Hence we are left with

$$\begin{aligned}
\mathfrak{C}_0^{(\beta)}(n,x) &= -2\sigma_1^{(\beta)}(n) + \sum_{|\gamma|<T} \frac{\sigma^{(\beta)}_{1-\rho/\beta}(n)}{\zeta'(\rho)}\frac{x^\rho}{\rho} + K(x,T) \\
&\quad + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(2\pi/x)^{2k}}{(2k)!k\zeta(2k+1)}\sigma^{(\beta)}_{1+2k/\beta}(n) \\
&\quad + \frac{1}{2\pi i}\left(\int_{c-iT}^{-\infty-iT} + \int_{-\infty+iT}^{c+iT}\right)\frac{\sigma^{(\beta)}_{1-s/\beta}(n)}{\zeta(s)}\frac{x^s}{s}ds - E_{1,T}(x) \\
&= -2\sigma_1^{(\beta)}(n) + \sum_{|\gamma|<T} \frac{\sigma^{(\beta)}_{1-\rho/\beta}(n)}{\zeta'(\rho)}\frac{x^\rho}{\rho} + K(x,T) \\
&\quad + \sum_{k=1}^{\infty} \frac{(-1)^{k-1}(2\pi/x)^{2k}}{(2k)!k\zeta(2k+1)}\sigma^{(\beta)}_{1+2k/\beta}(n) + E_{2,T}(x) - E_{1,T}(x),
\end{aligned}$$

where the last two terms are to be bounded. For the second integral, we split the range of integration in $(-\infty+iT, -1+iT) \cup (-1+iT, c+iT)$ and we write

$$\begin{aligned}
\int_{-\infty+iT}^{-1+iT} \frac{\sigma^{(\beta)}_{1-s/\beta}(n)}{\zeta(s)}\frac{x^s}{s}ds &= \int_{2+iT}^{\infty+iT} \frac{\sigma^{(\beta)}_{1-(1-s)/\beta}(n)}{\zeta(1-s)}\frac{x^{1-s}}{1-s}ds \\
&= \int_{2+iT}^{\infty+iT} \frac{x^{1-s}}{1-s}\frac{\sigma^{(\beta)}_{1-(1-s)/\beta}(n)2^{s-1}\pi^s}{\cos(\frac{\pi s}{2})\Gamma(s)}\frac{1}{\zeta(s)}ds \\
&\ll \int_2^\infty \frac{1}{T}\left(\frac{2\pi}{x}\right)^\sigma e^{((\beta+\sigma)\log n+\sigma-(\sigma-\frac{1}{2})\log\sigma}d\sigma \ll \frac{1}{Tx^2}.
\end{aligned}$$

We can now choose for each $\varepsilon > 0$, $T = T_\nu$ satisfying (4.11) and (4.12) such that

$$\frac{1}{\zeta(s)} \ll t^\varepsilon, \quad \frac{1}{2} \leqslant \sigma \leqslant 2, \quad t = T_\nu.$$

Thus the other part of the integral is

$$\begin{aligned}
\int_{-1+iT_\nu}^{c+iT_\nu} \frac{\sigma^{(\beta)}_{1-s/\beta}(n)}{\zeta(s)}\frac{x^s}{s}ds &\ll \int_{-1}^{\min(\beta,c)} T_\nu^{\varepsilon-1}e^{(\beta-\sigma+1)\log n}x^\sigma d\sigma + \int_{\min(\beta,c)}^{c} T_\nu^{\varepsilon-1}x^\sigma d\sigma \\
&\ll xT_\nu^{\varepsilon-1}.
\end{aligned}$$

The integral over $(2-iT_\nu, -\infty-iT_\nu)$ is dealt with similarly. It remains to bound $E_{1,T_\nu}(x)$, i.e. the three error terms on the right-hand side of (3.12.1) in [Tit86] . We have $\psi(q) = \sigma_1^{(\beta)}(n)$, $s = 0$, $c = 1 + \frac{1}{\log x}$ and $\alpha = 1$. Inserting these yields

$$\begin{aligned}
E_{T_\nu}(x) &= E_{2,T_\nu}(x) - E_{1,T_\nu}(x) \\
&\ll xT_\nu^{\varepsilon-1} + \frac{x\log x}{T_\nu} + \frac{x\sigma_1^\beta(n)\log x}{T_\nu} + \frac{x\sigma_1^\beta(n)}{T_\nu} \ll \frac{x\log x}{T_\nu^{1-\varepsilon}}.
\end{aligned}$$

If we assume that all non-trivial zeros are simple then term $K_{T_\nu}(x)$ disappears.

Theorem 4.1.1 can be illustrated by plotting the explicit formula as in Figure 4.1.

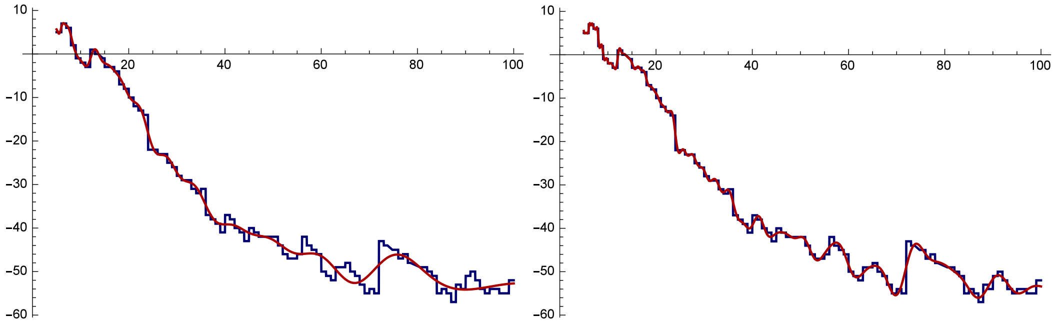

FIGURE 4.1: Plot of the partial sum $\mathfrak{C}^{\sharp,(1)}(12,x)$ in blue, while the main terms of Theorem 4.1.1 with 5 and 25 pairs of zeros and $5 \le x \le 100$ is shown in red.

Increasing the value of β does not affect the match. For $\beta = 2$, see Figure 4.2.

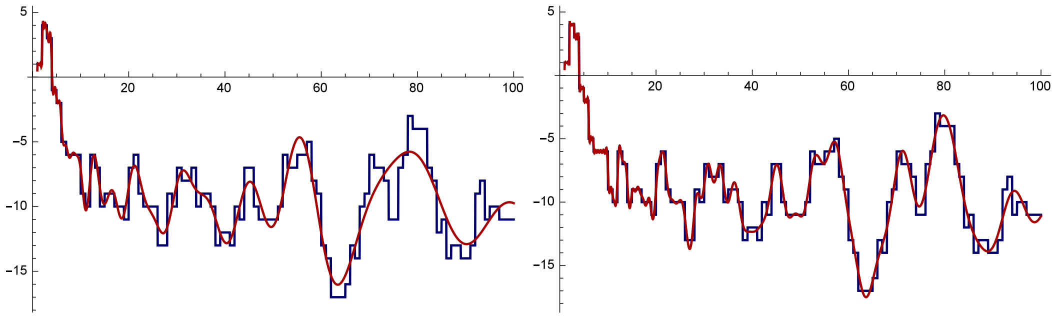

FIGURE 4.2: In blue the step function $\mathfrak{C}^{\sharp,(2)}(24,x)$ is plotted, while the main terms of Theorem 4.1.1 with 5 and 25 pairs of zeros and $1 \le x \le 100$ are plotted in red.

For $\beta = 3$ we have the same effect (Figure 4.3).

4.3 Proof of Theorem 4.1.2 and Corollary 4.1.1

We shall use the lemma in §3.12 of [Tit86]. Take $a_q = c_q^{(\beta)}(n)$, $\alpha = 1$ and let x be half an odd integer. Let $s = 1 + it$, then

$$\begin{aligned}\sum_{q<x} \frac{c_q^{(\beta)}(n)}{q^s} &= \frac{1}{2\pi i}\int_{c-iT}^{c+iT} \frac{\sigma^{(\beta)}_{1-(s+w)/\beta}(n)}{\zeta(s+w)}\frac{x^w}{w}dw + O\left(\frac{x^c}{Tc}\right) + O\left(\frac{1}{T}\sigma_1^{(\beta)}(n)\log x\right)\\ &= \frac{\sigma^{(\beta)}_{1-s/\beta}(n)}{\zeta(s)}\\ &\quad + \frac{1}{2\pi i}\left(\int_{c-iT}^{-\delta-iT} + \int_{-\delta-iT}^{-\delta+iT} + \int_{-\delta+iT}^{c+iT}\right)\frac{\sigma^{(\beta)}_{1-(s+w)/\beta}(n)}{\zeta(s+w)}\frac{x^w}{w}dw,\end{aligned}$$

where $c > 0$ and δ is small enough that $\zeta(s+w)$ has no zeros for

$$\mathrm{Re}(w) \geqslant -\delta, \quad |\mathrm{Im}(s+w)| = |t + \mathrm{Im}(w)| \leqslant |t| + T.$$

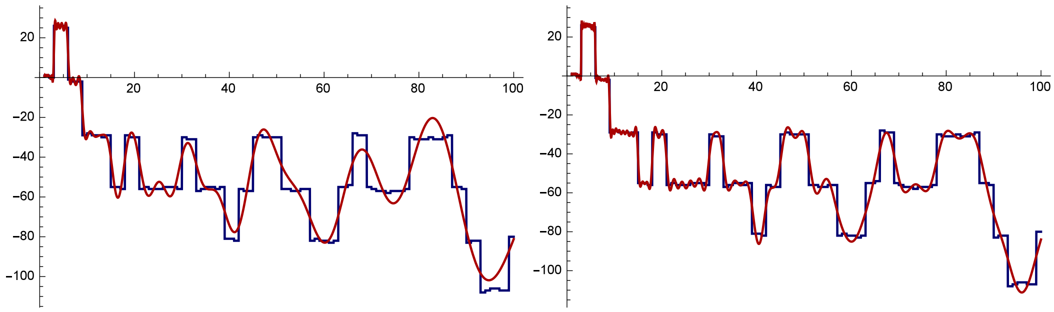

FIGURE 4.3: In blue the step function $\mathfrak{C}^{\sharp,(3)}(810, x)$ is plotted, while the main terms of Theorem 4.1.1 with 5 and 25 pairs of zeros and $1 \le x \le 100$ are plotted in red.

It is known from §3.6 of [Tit86] that $\zeta(s)$ has no zeros in the region $\sigma > 1 - A\log^{-9} t$, where A is a positive constant. Thus, we can take $\delta = A\log^{-9} T$. The contribution from the vertical integral is given by

$$\int_{-\delta-iT}^{-\delta+iT} \frac{\sigma^{(\beta)}_{1-(s+w)/\beta}(n)}{\zeta(s+w)} \frac{x^w}{w} dw \ll x^{-\delta} n^{\delta} \log^7 T \int_{-T}^{T} \frac{dv}{\sqrt{\delta^2 + v^2}} \ll x^{-\delta} n^{\delta} \log^8 T.$$

For the top horizontal integral we get

$$\begin{aligned}\int_{-\delta+iT}^{c+iT} \frac{\sigma^{(\beta)}_{1-(s+w)/\beta}(n)}{\zeta(s+w)} \frac{x^w}{w} dw &\ll \frac{\log^7 T}{T} \left(\int_{-\delta}^{\min(c,\beta-1)} n^{\beta-u} x^u du + \int_{\min(c,\beta-1)}^{c} x^u du \right) \\ &\ll \frac{\log^7 T}{T} x^c \left(\int_{-\delta}^{\min(c,\beta-1)} n^{\beta-u} du + c \right) \ll \frac{\log^7 T}{T} x^c n^{\delta},\end{aligned}$$

provided $x > 1$. For the bottom horizontal integral we proceed the same way. Consequently, we have the following

$$\sum_{q<x} \frac{c_q^{(\beta)}(n)}{q^s} - \frac{\sigma^{(\beta)}_{1-s/\beta}(n)}{\zeta(s)} \ll \frac{x^c}{Tc} + \frac{1}{T}\sigma_1^{(\beta)}(n)\log x + x^{-\delta} n^{\delta} \log^8 T + \frac{\log^7 T}{T} x^c n^{\delta}.$$

Now, we choose $c = 1/\log x$ so that $x^c = e$. We take $T = \exp\{(\log x)^{1/10}\}$ so that $\log T = (\log x)^{1/10}$, $\delta = A(\log x)^{-9/10}$ and $x^{\delta} = T^A$. Then it is seen that the right-hand side tends to zero as $x \to \infty$ and the result follows.

Corollary 4.1.1 follows by Lemma 4.2.4 for $\beta \ge 1$ and, by Theorem (4.1.2) with $\mathrm{Re}(s) \ge 1$. If $s = 1$ then the first equation follows. If in (4.4) we set $s = 1$ then the second equation follows. Setting $s = \beta = 1$ yields the third equation. Finally, putting $n = 1$ in the third equation yields the fourth equation.

The plots of Corollary 4.1.1 are illustrated below, in Figure 4.4 and Figure 4.5.

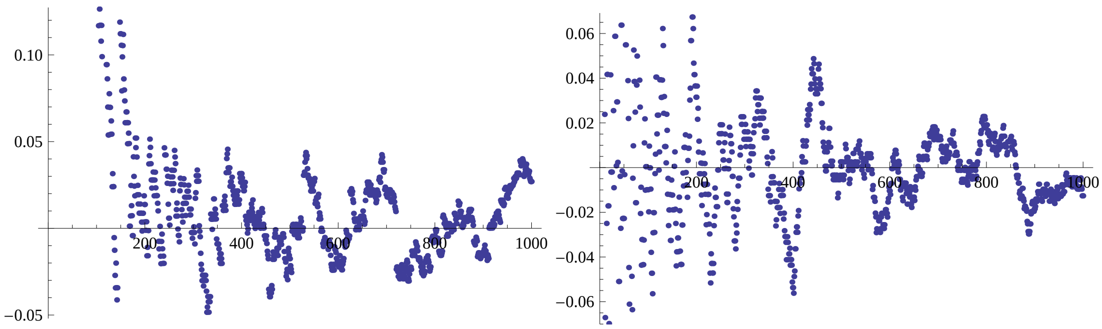

FIGURE 4.4: Plot of the partial sums $\sum_{q=1}^{x} c_q^{(1)}(24)/q$ and of $\sum_{q=1}^{x} c_q^{(2)}(24)/q$ for $1 \le x \le 1000$.

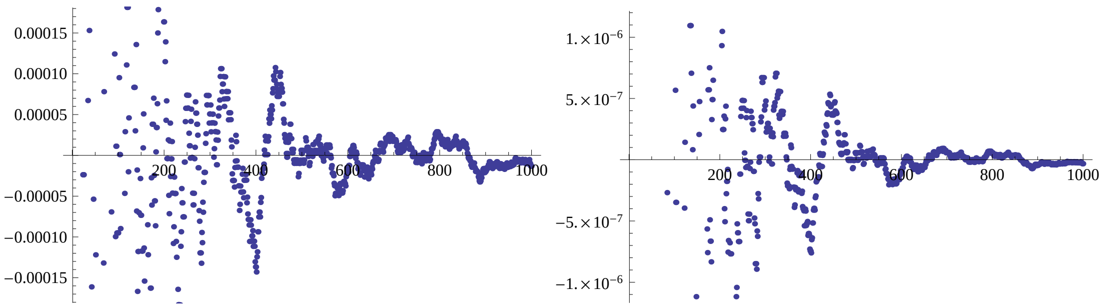

FIGURE 4.5: Plot of the partial sums $\sum_{q=1}^{x} c_q^{(2)}(24)/q^2 - \sigma_0^{(2)}(24)/\zeta(2)$ and of $\sum_{q=1}^{x} c_q^{(3)}(24)/q^3 - \sigma_0^{(3)}(24)/\zeta(3)$ for $1 \le x \le 1000$.

4.4 Proof of Theorem 4.1.3

In the lemma of §3.12 of [Tit86], take $a_q = c_q^{(\beta)}(n)$, $f(s) = \sigma_{1-s/\beta}^{(\beta)}(n)/\zeta(s)$, $c = 2$, and let x be half an odd integer. Then

$$\begin{aligned}
\sum_{q<x} \frac{c_q^{(\beta)}(n)}{q^s} &= \frac{1}{2\pi i}\int_{2-iT}^{2+iT} \frac{\sigma_{1-(s+w)/\beta}^{(\beta)}(n)}{\zeta(s+w)}\frac{x^w}{w}dw + O\left(\frac{x^2}{T}\right) \\
&= \frac{1}{2\pi i}\left(\int_{2-iT}^{\frac{1}{2}-\sigma+\delta-iT} + \int_{\frac{1}{2}-\sigma+\delta-iT}^{\frac{1}{2}-\sigma+\delta+iT} + \int_{\frac{1}{2}-\sigma+\delta+iT}^{2+iT}\right)\frac{\sigma_{1-(s+w)/\beta}^{(\beta)}(n)}{\zeta(s+w)}\frac{x^w}{w}dw \\
&\quad + \frac{\sigma_{1-s/\beta}^{(\beta)}(n)}{\zeta(s)} + O\left(\frac{x^2}{T}\right) \\
&= I_1 + I_2 + I_3 + \frac{\sigma_{1-s/\beta}^{(\beta)}(n)}{\zeta(s)} + O\left(\frac{x^2}{T}\right),
\end{aligned}$$

where $0 < \delta < \sigma - \frac{1}{2}$. If we assume RH, then $\zeta(s) \ll t^{\varepsilon}$ for $\sigma \ge \frac{1}{2}$ and $\forall \varepsilon > 0$ so that the first and third integrals are

$$I_1, I_3 \ll T^{-1+\varepsilon}\left(\int_{\frac{1}{2}-\sigma+\delta}^{\min(\beta-\sigma,2)} n^{\beta-\sigma+v}x^v dv + \int_{\min(\beta-\sigma,2)}^{2} x^v dv\right) \ll T^{-1+\varepsilon}x^2,$$

provided $x > 1$. The second integral is

$$I_2 \ll x^{\frac{1}{2}-\sigma+\delta} n^{\beta+\delta+1} \int_{-T}^{T} (1+|t|)^{-1+\varepsilon} dt \ll x^{\frac{1}{2}-\sigma+\delta} T^{\varepsilon}.$$

Thus we have

$$\sum_{q<x} \frac{c_q^{(\beta)}(n)}{q^s} = \frac{\sigma_{1-s/\beta}^{(\beta)}(n)}{\zeta(s)} + O(x^{\frac{1}{2}-\sigma+\delta} T^{\varepsilon}) + O(x^2 T^{\varepsilon-1}).$$

Taking $T = x^3$ the O-terms tend to zero as $x \to \infty$, and the result (4.7) follows. Conversely, if (4.7) is convergent for $\sigma > \frac{1}{2}$, then it is uniformly convergent for $\sigma \geq \sigma_0 > \frac{1}{2}$, and so in this region it represents an analytic function, which is $\sigma_{1-s/\beta}^{(\beta)}(n)/\zeta(s)$ for $\sigma > 1$ and so throughout the region $\sigma \geq \sigma_0 > \frac{1}{2}$. This means that the Riemann hypothesis is true and the proof is now complete.

4.5 Proof of Theorem 4.1.5

First, the Dirichlet series are given by the following result.

Lemma 4.5.1. *For* $\mathrm{Re}(s) > 1$ *and* $\beta, k \in \mathbb{N}$,

$$\sum_{n=1}^{\infty} \frac{\Lambda_{k,m}^{(\beta)}(n)}{n^s} = (-1)^k \sigma_{1-s/\beta}^{(\beta)}(m) \frac{\zeta^{(k)}(s)}{\zeta(s)},$$

where $\zeta^{(k)}(s)$ *is the* k^{th} *derivative of the Riemann zeta-function.*

Proof. Apply Lemma 4.2.4, and

$$\sum_{n=1}^{\infty} \frac{\log^k n}{n^s} = (-1)^k \zeta^{(k)}(s)$$

for $\mathrm{Re}(s) > 1$. The result follows by Dirichlet convolution. □

From Lemma 4.5.1 we deduce that

$$\Lambda_{1,m}^{(\beta)}(n) \ll n^{\varepsilon}$$

for each $\varepsilon > 0$, otherwise the sum would not be absolutely convergent for $\mathrm{Re}(s) > 1$. It is known (see for instance Lemma 12.2 of [MV07]) that for each real number $T \geq 2$ there is a T_1, $T \leq T_1 \leq T+1$, such that

$$\frac{\zeta'}{\zeta}(\sigma + iT_1) \ll (\log T)^2$$

uniformly for $-1 \leq \sigma \leq 2$. By using Perron's inversion formula with $\sigma_0 = 1 + 1/\log x$ we obtain

$$\psi_{0,m}^{(\beta)}(x) = -\frac{1}{2\pi i} \int_{\sigma_0 - iT_1}^{\sigma_0 + iT_1} \sigma_{1-s/\beta}^{(\beta)}(m) \frac{\zeta'}{\zeta}(s) \frac{x^s}{s} ds + R_1,$$

where

$$R_1 \ll \sum_{\substack{x/2<n<2x \\ n\neq x}} \Lambda_{1,m}^{(\beta)}(n) \min\left(1, \frac{x}{T|x-n|}\right) + \frac{x}{T}\sum_{n=1}^{\infty} \frac{\Lambda_{1,m}^{(\beta)}(n)}{n^{\sigma_0}} = R_{1,1} + \frac{x}{T}R_{1,2}. \quad (4.14)$$

The second sum $R_{1,2}$ is

$$-\sigma_{1-\sigma_0/\beta}^{(\beta)}(m)\frac{\zeta'}{\zeta}(\sigma_0) \asymp \frac{\sigma_{1-\sigma_0/\beta}^{(\beta)}(m)}{\sigma_0 - 1} = \sigma_{1-(1+1/\log x)/\beta}^{(\beta)}(m)\log x.$$

The term involving the generalized divisor function can be bounded in the following way:

$$\sigma_{1-(1+1/\log x)/\beta}^{(\beta)}(m) \leq m^{\beta+1/\log x}$$

if $\frac{1}{\log x} \leq \beta - 1$, and $\leq m$ otherwise. In both cases, this is bounded in x. For the first sum $R_{1,1}$ we do as follows. The terms for which $x+1 \leq n < 2x$ contribute an amount $R_{1,1,1}$ which is

$$R_{1,1,1} \ll \sum_{x+1\leqslant n<2x} \frac{x^{1+\varepsilon}}{T(n-x)} \ll \frac{x^{1+\varepsilon}\log x}{T}.$$

The terms for which $x/2 < n \leq x-1$ are dealt with in a similar way. The remaining terms for which $x-1 < n < x+1$ contribute an amount $R_{1,1,2}$ which is

$$R_{1,1,2} \ll x^{\varepsilon}\min\left(1, \frac{x}{T\langle x\rangle_\beta}\right),$$

therefore, the final bound for R_1 in (4.14) is

$$R_1 \ll x^{\varepsilon}\min\left(1, \frac{x}{T\langle x\rangle_\beta}\right) + \frac{x^{1+\varepsilon}\log x}{T}.$$

We denote an odd positive integer by N and the contour consisting of line segments connecting $\sigma_0 - iT_1, -N - iT_1, -N + iT_1, \sigma_0 + iT_1$ by $\mathcal{D}$. An application of Cauchy's residue theorem yields

$$\psi_{0,m}^{(\beta)}(x) = M_0 + M_1 + M_\rho + M_{-2k} + R_1 + R_2$$

where the terms on the right-hand sides are the residues at $s=0$, $s=1$, the non-trivial zeros ρ and at the trivial zeros $-2k$ for $k = 1, 2, 3, \cdots$, respectively, and where

$$R_2 = -\frac{1}{2\pi i}\oint_{\mathcal{D}} \sigma_{1-s/\beta}^{(\beta)}(m)\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}ds.$$

For the constant term we have

$$M_0 = \operatorname*{Res}_{s=0} \sigma_{1-s/\beta}^{(\beta)}(m)\frac{\zeta'}{\zeta}(s)\frac{x^s}{s} = \sigma_1^{(\beta)}(m)\frac{\zeta'}{\zeta}(0) = \sigma_1^{(\beta)}(m)\log(2\pi),$$

and for the leading term

$$M_1 = \operatorname*{Res}_{s=1} \sigma_{1-s/\beta}^{(\beta)}(m)\frac{\zeta'}{\zeta}(s)\frac{x^s}{s} = \sigma_{1-1/\beta}^{(\beta)}(m)x.$$

The fluctuaring term coming from the non-trivial zeros yields

$$M_\rho = \sum_\rho \operatorname*{Res}_{s=\rho} \sigma^{(\beta)}_{1-s/\beta}(m)\frac{\zeta'}{\zeta}(s)\frac{x^s}{s} = \sum_\rho \sigma^{(\beta)}_{1-\rho/\beta}(m)\frac{x^\rho}{\rho},$$

by the use of the logarithmic derivative of the Hadamard product of the Riemann zeta-function, and finally for the trivial zeros

$$M_{-2k} = \sum_{k=1}^{\infty} \operatorname*{Res}_{s=-2k} \sigma^{(\beta)}_{1-s/\beta}(m)\frac{\zeta'}{\zeta}(s)\frac{x^s}{s} = \sum_{k=1}^{\infty} \sigma^{(\beta)}_{1+2k/\beta}(m)\frac{x^{-2k}}{-2k}.$$

Since $|\sigma \pm iT_1| \geqslant T$, we see, by our choice of T_1, that

$$\begin{aligned}\int_{-1\pm iT_1}^{\sigma_0\pm iT_1} \sigma^{(\beta)}_{1-s/\beta}(m)\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}ds &\ll \frac{\log^2 T}{T}\left(\int_{-1}^{\min(\beta,\sigma_0)} \left(\frac{x}{m}\right)^\sigma d\sigma + \int_{\min(\beta,\sigma_0)}^{\sigma_0} x^\sigma d\sigma\right)\\ &\ll \frac{x\log^2 T}{T\log x} \ll \frac{x\log^2 T}{T}.\end{aligned}$$

Next, we invoke the following result (see Lemma 12.4 of [MV07]): if $\mathcal{A}$ denotes the set of points $s \in \mathbb{C}$ such that $\sigma \leq -1$ and $|s+2k| \geq 1/4$ for every positive integer k, then

$$\frac{\zeta'}{\zeta}(s) \ll \log(|s|+1)$$

uniformly for $s \in \mathcal{A}$. This, combined with the fact that

$$\frac{\log|\sigma \pm iT_1|}{|\sigma \pm iT_1|} \ll \frac{\log T}{T},$$

gives us

$$\int_{-N\pm iT_1}^{-1\pm iT_1} \sigma^{(\beta)}_{1-s/\beta}(m)\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}ds \ll \frac{\log T}{T}\int_{-\infty}^{-1}\left(\frac{x}{m}\right)^\sigma d\sigma \ll \frac{\log T}{xT\log x} \ll \frac{\log T}{T}.$$

Thus this bounds the horizontal integrals. Finally, for the left vertical integral, we have that $|-N+iT| \geq N$ and by the above result regarding the bound of the logarithmic derivative we also see that

$$\begin{aligned}\int_{-N-iT_1}^{-N+iT_1} \sigma^{(\beta)}_{1-s/\beta}(m)\frac{\zeta'}{\zeta}(s)\frac{x^s}{s}ds &\ll \frac{\log NT}{N}x^{-N}\sigma^{(\beta)}_{1+N/\beta}(m)\int_{-T_1}^{T_1} dt\\ &\ll \frac{T\log NT}{N}\left(\frac{m}{x}\right)^N = o(1)\end{aligned}$$

as $N \to \infty$ since $x > m$.

4.6 Proof of Theorems 4.1.6 and 4.1.7

Let us denote by $\rho = \beta^* + i\gamma$ a non-trivial zero of the Riemann zeta-function. For this, we will use the result that if $|\gamma| < T$, where T is large, then $\beta^* < 1 - c_1/\log T$, where c_1

is a positive absolute constant. This immediately yields

$$|x^\rho| = x^{\beta^*} < xe^{-c_1 \log x/\log T}.$$

Moreover, $|\rho| \geq \gamma$, for $\gamma > 0$. We recall that the number of zeros $N(t)$ up to height t is (Chapter 18, p. 111 of [Dav66].)

$$N(t) = \frac{t}{2\pi}\log\frac{t}{2\pi} - \frac{t}{2\pi} + O(\log t) \ll t\log t.$$

We need to estimate the following sum

$$\sum_{0<\gamma<T} \frac{\sigma^{(\beta)}_{1-\gamma/\beta}(m)}{\gamma} = \sum_{1<\gamma<T} \frac{\sigma^{(\beta)}_{1-\gamma/\beta}(m)}{\gamma}.$$

This is

$$\sum_{1<\gamma<T} \frac{\sigma^{(\beta)}_{1-\gamma/\beta}(m)}{\gamma} \ll \int_1^T \frac{\sigma^{(\beta)}_{1-t/\beta}(m)}{t^2} N(t)dt \ll m\int_1^\beta \frac{\log t}{t}dt + m^{\beta+1}\int_\beta^T \frac{\log t}{tm^t}dt \ll \log^2 T. \tag{4.15}$$

Therefore,

$$\sum_{|\gamma|<T} \left|\sigma^{(\beta)}_{1-\rho/\beta}(m)\frac{x^\rho}{\rho}\right| \ll x(\log T)^2 e^{-c_1\log x/\log T}. \tag{4.16}$$

Without loss of generality we take x to be an integer in which case the error term of the explicit formula of Theorem 4.1.5 becomes

$$R(x,T) \ll \frac{x^{1+\varepsilon}\log x}{T} + \frac{x\log^2 T}{T}. \tag{4.17}$$

Finally, we can bound the sum

$$\sum_{k=1}^\infty \sigma^{(\beta)}_{1+2k/\beta}(m)\frac{x^{-2k}}{2k} \leq m^{\beta+1}\sum_{k=1}^\infty m^{2k}\frac{x^{-2k}}{2k} = \frac{1}{2}m^{\beta+1}\log\left(1-\left(\frac{x}{m}\right)^{-2}\right) = o(1). \tag{4.18}$$

Thus, using Theorem 4.1.5 and (4.16), (4.17), (4.18), we have

$$|\psi^{(\beta)}_m(x) - \sigma^{(\beta)}_{1-1/\beta}(m)x| \ll \frac{x^{1+\varepsilon}\log x}{T} + \frac{x\log^2 T}{T} + x(\log T)^2 e^{-c_1\log x/\log T},$$

for large x. Let us now take T as a function of x by setting $(\log T)^2 = \log x$ so that

$$\begin{aligned}|\psi^{(\beta)}_m(x) - \sigma^{(\beta)}_{1-1/\beta}(m)x| &\ll x^{1+\varepsilon}\log x e^{-(\log x)^{1/2}} + x(\log x)e^{-c_1(\log x)^{1/2}} \\ &\ll x^{1+\varepsilon}e^{-c_2(\log x)^{1/2}},\end{aligned}$$

for all $\varepsilon > 0$ provided that c_2 is a suitable constant that is less than both 1 and c_1.

Next, if we assume the Riemann hypothesis, then $|x^\rho| = x^{1/2}$ and (4.15) stays the same. Thus, the explicit formula yields

$$|\psi^{(\beta)}_m(x) - \sigma^{(\beta)}_{1-1/\beta}(m)x| = O\left(x^{1/2}\log^2 T + \frac{x^{1+\varepsilon}\log x}{T} + \frac{x\log^2 T}{T}\right)$$

provided that x is an integer. Taking $T = x^{1/2}$ leads to

$$\psi_m^{(\beta)}(x) = \sigma_{1-1/\beta}^{(\beta)}(m)x + O(x^{1/2}\log^2 x + x^{1/2+\varepsilon}\log x) = \sigma_{1-1/\beta}^{(\beta)}(m)x + O(x^{1/2+\varepsilon}).$$

4.7 Proof of Theorem 4.1.8

We now look at the contour integral

$$\Upsilon^{(\beta)}(n,z) = \oint_{\Omega} \frac{\sigma_{1-s/\beta}^{(\beta)}(n)}{\zeta(s)} e^{sz} ds$$

taken around the path $\Omega = [-1/2, 3/2, 3/2 + iT_n, -1/2 + iT_n]$.

For the upper horizontal integral we have

$$\begin{aligned}
\left| \int_{-1/2+iT_m}^{3/2+iT_m} \frac{\sigma_{1-s/\beta}^{(\beta)}(n)}{\zeta(s)} e^{sz} ds \right| &\leqslant \int_{-1/2}^{\min(\beta,3/2)} \left| \frac{\sigma_{1-s/\beta}^{(\beta)}(n) e^{sz}}{\zeta(\sigma + iT_m)} \right| d\sigma \\
&\quad + \int_{\min(\beta,3/2)}^{3/2} \left| \frac{\sigma_{1-s/\beta}^{(\beta)}(n) e^{sz}}{\zeta(\sigma + iT_m)} \right| d\sigma \\
&\ll T_m^{c_1} n^{\beta+1} e^{-T_m y} \int_{-1/2}^{\min(\beta,3/2)} n^{-\sigma} e^{\sigma x} d\sigma \\
&\quad + n e^{-T_m y} \int_{\min(\beta,3/2)}^{3/2} e^{\sigma x} d\sigma \\
&\to 0,
\end{aligned}$$

as $m \to \infty$. An application of Cauchy's residue theorem yields

$$\begin{aligned}
&\int_{-1/2+i\infty}^{-1/2} \frac{\sigma_{1-s/\beta}^{(\beta)}(n)}{\zeta(s)} e^{sz} ds + \int_{-1/2}^{3/2} \frac{\sigma_{1-s/\beta}^{(\beta)}(n)}{\zeta(s)} e^{sz} ds + \int_{3/2}^{3/2+i\infty} \frac{\sigma_{1-s/\beta}^{(\beta)}(n)}{\zeta(s)} e^{sz} ds \\
&= 2\pi i \varpi_n^{(\beta)}(z),
\end{aligned} \tag{4.19}$$

where for $\mathrm{Im}(z) > 0$ we have

$$\varpi_n^{(\beta)}(z) = \lim_{m\to\infty} \sum_{\substack{\rho \\ 0<\mathrm{Im}\,\rho<T_m}} \frac{1}{(k_\rho - 1)!} \frac{d^{k_\rho - 1}}{ds^{k_\rho - 1}} \left[(s-\rho)^{k_\rho} \frac{\sigma_{1-s/\beta}^{(\beta)}(n)}{\zeta(s)} e^{sz} \right]_{s=\rho}$$

with k_ρ denoting the order of multiplicity of the non-trivial zero ρ of the Riemann zeta-function. We denote by $\varpi_{1,n}^{(\beta)}(z)$ and by $\varpi_{2,n}^{(\beta)}(z)$ the first and second integrals on the left hand-side of (4.19) respectively. If we operate under the assumption that there are no multiple zeros, then the above can be simplified to (4.9). This is done for the sake of simplicity, since dealing with this extra term would relax this assumption.

If $z \in \mathbb{H}$ then by (4.19) one has

$$2\pi i \varpi_n^{(\beta)}(z) = \varpi_{1,n}^{(\beta)}(z) + \varpi_{2,n}^{(\beta)}(z) + \varpi_{3,n}^{(\beta)}(z), \tag{4.20}$$

where the last term is given by the vertical integral on the right of the Ω contour

$$\varpi_{3,n}^{(\beta)}(z)=\int_{3/2}^{3/2+i\infty}\frac{\sigma_{1-s/\beta}^{(\beta)}(n)}{\zeta(s)}e^{sz}ds.$$

By the use of the Dirichlet series of $c_q^{(\beta)}(n)$ given in Lemma 4.2.4 and since we are in the region of absolute convergence we see that

$$\varpi_{3,n}^{(\beta)}(z)=\sum_{q=1}^{\infty}c_q^{(\beta)}(n)\int_{3/2}^{3/2+i\infty}e^{sz-s\log q}ds=-e^{3z/2}\sum_{q=1}^{\infty}\frac{c_q^{(\beta)}(n)}{q^{3/2}(z-\log q)}.$$

By standard bounds of Stirling and the functional equation of the Riemann zeta-function, we have (see [Tit86])

$$|\zeta(-\tfrac{1}{2}+it)|\approx(1+|t|)$$

as $|t|\to\infty$. Therefore, we see that

$$\begin{aligned}|\varpi_{1,n}^{(\beta)}(z)|&=\left|\int_{-1/2+i\infty}^{-1/2}\frac{\sigma_{1-s/\beta}^{(\beta)}(n)}{\zeta(s)}e^{sz}ds\right|=O\left(n^{\beta+3/2}e^{-x/2}\int_0^{\infty}e^{-ty}dt\right)\\&=O\left(\frac{n^{\beta+3/2}e^{-x/2}}{y}\right),\end{aligned}$$

and $\varpi_{1,n}^{(\beta)}(z)$ is absolutely convergent for $y=\mathrm{Im}(z)>0$. We know that $\varpi_n^{(\beta)}(z)$ is analytic for $y>0$ and the next step is to show that that it can be meromorphically continued for $y>-\pi$. To this end, we go back to the integral

$$\varpi_{1,n}^{(\beta)}(z)=-\int_{-1/2}^{-1/2+i\infty}\frac{\sigma_{1-s/\beta}^{(\beta)}(n)}{\zeta(s)}e^{sz}ds$$

with $y>0$. The functional equation of $\zeta(s)$ yields

$$\begin{aligned}\varpi_{1,n}^{(\beta)}(z)&=-\int_{-1/2}^{-1/2+i\infty}\sigma_{1-s/\beta}^{(\beta)}(n)\frac{\Gamma(s)}{\zeta(1-s)}e^{s(z-\log 2\pi-i\pi/2)}ds\\&\quad-\int_{-1/2}^{-1/2+i\infty}\sigma_{1-s/\beta}^{(\beta)}(n)\frac{\Gamma(s)}{\zeta(1-s)}e^{s(z-\log 2\pi+i\pi/2)}ds\\&=\varpi_{11,n}^{(\beta)}(z)+\varpi_{12,n}^{(\beta)}(z).\end{aligned}\tag{4.21}$$

Since one has by the Stirling bound that

$$\frac{\Gamma(-\frac{1}{2}+it)}{\zeta(\frac{3}{2}-it)}\ll e^{-\pi t/2}$$

it then follows that

$$\varpi_{11,n}^{(\beta)}(z)\ll n^{\beta+3/2}\int_0^{\infty}e^{-\pi t/2}e^{-\frac{1}{2}x-ty+t\pi/2}dt\ll\frac{e^{-x/2}n^{\beta+3/2}}{y},$$

and hence $\varpi_{11,n}^{(\beta)}(z)$ is regular for $y > 0$. Similarly,

$$\varpi_{12,n}^{(\beta)}(z) \ll n^{\beta+3/2}e^{-x/2}\int_0^\infty e^{-(\pi+y)t}dt \ll \frac{n^{\beta+3/2}e^{-x/2}}{y+\pi},$$

so that $\varpi_{12,n}^{(\beta)}(z)$ is regular for $y > -\pi$. Let us further split $\varpi_{11,n}^{(\beta)}(z)$

$$\begin{aligned}\varpi_{11,n}^{(\beta)}(z) &= \left(-\int_{-1/2-i\infty}^{-1/2+i\infty} + \int_{-1/2-i\infty}^{-1/2}\right)e^{s(z-\log 2\pi - i\pi/2)}\sigma_{1-s/\beta}^{(\beta)}(n)\frac{\Gamma(s)}{\zeta(1-s)}ds \\ &= I_{1,n}^{(\beta)}(z) + I_{2,n}^{(\beta)}(z).\end{aligned}$$

By the same technique as above, it follows that the integral $I_{2,n}^{(\beta)}(z)$ is convergent for $y < \pi$. Moreover, since $\varpi_{11,n}^{(\beta)}(z)$ is regular for $y > 0$, then it must be that $I_{1,n}^{(\beta)}(z)$ is convergent for $0 < y < \pi$. Let

$$f(n,q,s,z) = \sigma_{1-s/\beta}^{(\beta)}(n)e^{s(z-\log 2\pi - i\pi/2+\log q)}\Gamma(s).$$

By the theorem of residues we see that

$$\begin{aligned}-\int_{-1/2-i\infty}^{-1/2+i\infty} f(n,q,s,z)ds &= -\int_{1-i\infty}^{1+i\infty} f(n,q,s,z)ds + 2\pi i \operatorname*{res}_{s=0} f(n,q,s,z) \\ &= -\int_{1-i\infty}^{1+i\infty} f(n,q,s,z)ds + 2\pi i \sigma_1^{(\beta)}(n). \end{aligned} \tag{4.22}$$

This last integral integral is equal to

$$\int_{1-i\infty}^{1+i\infty} f(n,q,s,z)ds = 2\pi i \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(e^{-z}\frac{2\pi i}{q}\right)^k \sigma_{1+k/\beta}^{(\beta)}(n), \tag{4.23}$$

where the last sum is absolutelty convergent. To prove this note that

$$\mathrm{Re}(e^{-(z-\log 2\pi - i\pi/2+\log q)}) = (e^{-x}2\pi/q)\sin y > 0$$

for $0 < y < \pi$. Next, consider the path of integration with vertices $[1\pm iT]$ and $[-N\pm iT]$, where N is an odd positive integer. By Cauchy's residue theorem

$$\begin{aligned}&\left(\int_{1-iT}^{1+iT} - \int_{-N+iT}^{1+iT} - \int_{-N-iT}^{-N+iT} + \int_{-N-iT}^{1-iT}\right)\left(e^{-z}\frac{2\pi i}{q}\right)^{-s}\sigma_{1-s/\beta}^{(\beta)}(n)\Gamma(s)ds \\ &= 2\pi i\sum_{k=0}^{N-1}\frac{(-1)^k}{k!}\left(e^{-z}\frac{2\pi i}{q}\right)^k\sigma_{1+k/\beta}^{(\beta)}(n).\end{aligned}$$

The third integral on the far left of the path can be bounded in the following way

$$\begin{aligned}I_3 &:= \int_{-N-iT}^{-N+iT}\left(e^{-z}\frac{2\pi i}{q}\right)^{-s}\sigma_{1-s/\beta}^{(\beta)}(n)\Gamma(s)ds \\ &\ll \int_{-T}^{T}\left(e^{-x}\frac{2\pi}{q}\right)^N e^{-t(y-\frac{\pi}{2})}n^{\beta+N+1}e^{-\frac{\pi}{2}|t|}dt\end{aligned}$$

$$\ll \left(e^{-x}\frac{2\pi n}{q}\right)^N \int_{-T}^{T} e^{-t(y-\frac{\pi}{2})}e^{-\frac{\pi}{2}|t|}dt \ll \left(e^{-x}\frac{2\pi n}{q}\right)^N (e^{T(y-\pi)} + e^{-Ty})$$
$$\ll e^{-Nx+N\log\frac{2\pi n}{q}}e^{-T\min(y,\pi-y)}.$$

We now bound the horizontal parts. For the top one,

$$\begin{aligned} I_+ &:= \int_{-N+iT}^{1+iT} \left(e^{-z}\frac{2\pi i}{q}\right)^{-s} \sigma^{(\beta)}_{1-s/\beta}(n)\Gamma(s)ds \\ &\ll \int_{-N}^{1} \left(e^{-x}\frac{2\pi}{q}\right)^{-\sigma} e^{-T(y-\frac{\pi}{2})}n^{\beta-\sigma+1}T^{\frac{1}{2}}e^{-T\frac{\pi}{2}}d\sigma \\ &\ll T^{\frac{1}{2}}e^{-Ty}\int_{-N}^{1}\left(e^{-x}\frac{2\pi n}{q}\right)^{-\sigma} d\sigma \ll T^{\frac{3}{2}}e^{T(y-\pi)}\left(e^{-x}\frac{2\pi n}{q}\right)^N \\ &\ll T^{\frac{1}{2}}e^{-Nx+N\log\frac{2\pi n}{q}}e^{-Ty}, \end{aligned}$$

and analogously for the bottom one

$$\begin{aligned} I_- &:= \int_{-N-iT}^{1-iT} \left(e^{-z}\frac{2\pi i}{q}\right)^{-s} \sigma^{(\beta)}_{1-s/\beta}(n)\Gamma(s)ds \\ &\ll \int_{-N}^{1} \left(e^{-x}\frac{2\pi}{q}\right)^{-\sigma} e^{T(y-\frac{\pi}{2})}n^{\beta-\sigma+1}T^{\frac{1}{2}}e^{-T\frac{\pi}{2}}d\sigma \\ &\ll T^{\frac{1}{2}}e^{-T(\pi-y)}\left(e^{-x}\frac{2\pi n}{q}\right)^N \ll T^{\frac{1}{2}}e^{-Nx+N\log\frac{2\pi n}{q}}e^{-T(\pi-y)}. \end{aligned}$$

Now, let $T = T(N)$ be such that

$$T > \frac{N(-x+\log\frac{2\pi n}{q})}{\min(y,\pi-y)}.$$

It is now easy to see that all of the three parts tend to 0 as $N\to\infty$ through odd integers, and thus the result follows. Thus, putting together (4.23) with (4.21) and (4.22) gives us

$$\begin{aligned} I^{(\beta)}_{1,n}(z) &= -\int_{-1/2-i\infty}^{-1/2+i\infty} e^{s(z-\log 2\pi - i\pi/2)}\sigma^{(\beta)}_{1-s/\beta}(n)\frac{\Gamma(s)}{\zeta(1-s)}ds \\ &= -\sum_{q=1}^{\infty}\frac{\mu(q)}{q}\int_{-1/2-i\infty}^{-1/2+i\infty} e^{s(z-\log 2\pi-i\pi/2)}q^s\sigma^{(\beta)}_{1-s/\beta}(n)\Gamma(s)ds \\ &= -\sum_{q=1}^{\infty}\frac{\mu(q)}{q}\left(2\pi i\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(e^{-z}\frac{2\pi i}{q}\right)^k\sigma^{(\beta)}_{1+k/\beta}(n) - 2\pi i\sigma^{(\beta)}_1(n)\right) \\ &= -2\pi i\sum_{q=1}^{\infty}\frac{\mu(q)}{q}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(e^{-z}\frac{2\pi i}{q}\right)^k\sigma^{(\beta)}_{1+k/\beta}(n) \end{aligned} \tag{4.24}$$

since $\sum_{q=1}^{\infty}\mu(q)/q = 0$. Moreover,

$$|(2\pi i)^{-1}I^{(\beta)}_{1,n}(z)| = \left|\sum_{q=1}^{\infty}\frac{\mu(q)}{q}\left(\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(e^{-z}\frac{2\pi i}{q}\right)^k\sigma^{(\beta)}_{1+k/\beta}(n) - \sigma^{(\beta)}_1(n)\right)\right|$$

$$
\begin{aligned}
&= \left| \sum_{q=1}^{\infty} \frac{\mu(q)}{q} \left(\sum_{k=1}^{\infty} \frac{(-1)^k}{k!} \left(e^{-z} \frac{2\pi i}{q} \right)^k \sigma_{1+k/\beta}^{(\beta)}(n) \right) \right| \\
&\leqslant n^{\beta+1} \sum_{q=1}^{\infty} \frac{1}{q} \left(\sum_{k=1}^{\infty} \frac{1}{k!} \left(e^{-x} \frac{2\pi}{q} \right)^k n^k \right) \\
&= n^{\beta+1} \sum_{q=1}^{\infty} \frac{1}{q} \left(\exp\left(e^{-x} \frac{2\pi}{q} n \right) - 1 \right) \\
&\ll n^{\beta+1} e^{2\pi n/e^x} \sum_{q \leqslant [2\pi n/e^x]} \frac{1}{q} + \frac{2\pi n^{\beta+2}}{e^x} \sum_{q \geqslant [2\pi n/e^x]+1} \frac{1}{q^2} \ll c_2(x),
\end{aligned}
$$

and the series on the right hand-side of (4.24) is absolutely convergent for all y. Thus, this proves the analytic continuation of $\varpi_{1,n}^{(\beta)}(z)$ to $y > -\pi$. For $|y| < \pi$,

$$
\begin{aligned}
\varpi_{1,n}^{(\beta)}(z) &= I_{1,n}^{(\beta)}(z) + I_{2,n}^{(\beta)}(z) + \varpi_{12,n}^{(\beta)}(z) \\
&= -2\pi i \sum_{q=1}^{\infty} \frac{\mu(q)}{q} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \left(e^{-z} \frac{2\pi i}{q} \right)^k \sigma_{1+k/\beta}^{(\beta)}(n) \\
&\quad + \int_{-1/2-i\infty}^{-1/2} e^{s(z-\log 2\pi - i\pi/2)} \sigma_{1-s/\beta}^{(\beta)}(n) \frac{\Gamma(s)}{\zeta(1-s)} ds \\
&\quad - \int_{-1/2}^{-1/2+i\infty} e^{s(z-\log 2\pi + i\pi/2)} \sigma_{1-s/\beta}^{(\beta)}(n) \frac{\Gamma(s)}{\zeta(1-s)} ds
\end{aligned} \tag{4.25}
$$

where the first term is holomorphic for all y, the second one for $y < \pi$ and the third for $y > -\pi$. Hence, this last equation shows the continuation of $\varpi_n^{(\beta)}(z)$ to the region $y > -\pi$. To complete the proof of the theorem, consider the function

$$
\hat{\varpi}_n^{(\beta)}(z) = \lim_{m\to\infty} \sum_{\substack{\rho \\ -T_m < \operatorname{Im}\rho < 0}} \frac{\sigma_{1-\rho/\beta}^{(\beta)}(n)}{\zeta'(\rho)} e^{\rho z},
$$

where the zeros are in the lower part of the critical strip and z now belongs to the lower half-plane $\hat{\mathbb{H}} = \{z \in \mathbb{C} : \operatorname{Im}(z) < 0\}$. It then follows by repeating the above argument that

$$
\hat{\varpi}_{1,n}^{(\beta)}(z) = \hat{\varpi}_{11,n}^{(\beta)}(z) + \hat{\varpi}_{12,n}^{(\beta)}(z),
$$

where

$$
\hat{\varpi}_{11,n}^{(\beta)}(z) = -\int_{-1/2-i\infty}^{-1/2} e^{s(z-\log 2\pi - i\pi/2)} \sigma_{1-s/\beta}^{(\beta)}(n) \frac{\Gamma(s)}{\zeta(1-s)} ds
$$

is absolutely convergent for $y < \pi$ and

$$
\hat{\varpi}_{12,n}^{(\beta)}(z) = -\int_{-1/2-i\infty}^{-1/2} e^{s(z-\log 2\pi + i\pi/2)} \sigma_{1-s/\beta}^{(\beta)}(n) \frac{\Gamma(s)}{\zeta(1-s)} ds
$$

is absolutely convergent for $y < 0$. Spliting up the first integral just as before and using a similar analysis to the one we have just carried out, but using the fact that $\zeta(\bar{s}) = \overline{\zeta(s)}$

and choosing T_m ($m \le T_m \le m+1$) such that

$$\left|\frac{1}{\zeta(\sigma - iT_n)}\right| < T_n^{c_1}, \quad -1 \leqslant \sigma \leqslant 2,$$

yields

$$\begin{aligned}\hat{\varpi}_{1,n}^{(\beta)}(z) = &-2\pi i \sum_{q=1}^{\infty} \frac{\mu(q)}{q} \sum_{k=0}^{\infty} \frac{1}{k!}\left(e^{-z}\frac{2\pi i}{q}\right)^k \sigma_{1+k/\beta}^{(\beta)}(n)\\ &- \int_{-1/2-i\infty}^{-1/2} e^{s(z-\log 2\pi - i\pi/2)} \sigma_{1-s/\beta}^{(\beta)}(n) \frac{\Gamma(s)}{\zeta(1-s)} ds\\ &+ \int_{-1/2}^{-1/2+i\infty} e^{s(z-\log 2\pi + i\pi/2)} \sigma_{1-s/\beta}^{(\beta)}(n) \frac{\Gamma(s)}{\zeta(1-s)} ds. \end{aligned} \tag{4.26}$$

Therefore, $\hat{\varpi}_n^{(\beta)}(z)$ admits an analytic continuation from $y < 0$ to the half-plane $y < \pi$.

4.8 Proof of Theorem 4.1.9 and Theorem 4.1.10

Adding up the two results (4.25) and (4.26) of our previous section

$$\varpi_{1,n}^{(\beta)}(z) + \hat{\varpi}_{1,n}^{(\beta)}(z) = -2\pi i \sum_{q=1}^{\infty} \frac{\mu(q)}{q} \sum_{k=0}^{\infty} \frac{1}{k!}\left\{\left(e^{-z}\frac{2\pi i}{q}\right)^k + \left(-e^{-z}\frac{2\pi i}{q}\right)^k\right\}\sigma_{1+k/\beta}^{(\beta)}(n).$$

The other terms in (4.20) do not contribute since

$$\varpi_{2,n}^{(\beta)}(z) + \hat{\varpi}_{2,n}^{(\beta)}(z) = \left(\int_{-1/2}^{3/2} + \int_{3/2}^{-1/2}\right) e^{sz} \frac{\sigma_{1-s/\beta}^{(\beta)}(n)}{\zeta(s)} ds = 0,$$

and by the Theorem 4.1.8 we have

$$\varpi_{3,n}^{(\beta)}(z) + \hat{\varpi}_{3,n}^{(\beta)}(z) = 0.$$

Consequently, we have

$$\varpi_n^{(\beta)}(z) + \hat{\varpi}_n^{(\beta)}(z) = -\sum_{q=1}^{\infty} \frac{\mu(q)}{q} \sum_{k=0}^{\infty} \frac{1}{k!}\left\{\left(e^{-z}\frac{2\pi i}{q}\right)^k + \left(-e^{-z}\frac{2\pi i}{q}\right)^k\right\}\sigma_{1+k/\beta}^{(\beta)}(n) \tag{4.27}$$

for $|y| < \pi$. Thus, once again, by the previous theorem for all $y < \pi$ by analytic continuation

$$\varpi_n^{(\beta)}(z) = -\hat{\varpi}_n^{(\beta)}(z) - \sum_{q=1}^{\infty} \frac{\mu(q)}{q} \sum_{k=0}^{\infty} \frac{1}{k!}\left\{\left(e^{-z}\frac{2\pi i}{q}\right)^k + \left(-e^{-z}\frac{2\pi i}{q}\right)^k\right\}\sigma_{1+k/\beta}^{(\beta)}(n),$$

and for $y > -\pi$

$$\hat{\varpi}_n^{(\beta)}(z) = -\varpi_n^{(\beta)}(z) - \sum_{q=1}^{\infty} \frac{\mu(q)}{q} \sum_{k=0}^{\infty} \frac{1}{k!}\left\{\left(e^{-z}\frac{2\pi i}{q}\right)^k + \left(-e^{-z}\frac{2\pi i}{q}\right)^k\right\}\sigma_{1+k/\beta}^{(\beta)}(n).$$

This shows that $\varpi_n^{(\beta)}(z)$ and $\hat{\varpi}_n^{(\beta)}(z)$ can be analytically continued over $\mathbb{C}$ as meromorphic functions and that (4.27) holds for all z. To prove the functional equation, we look at the zeros. If ρ is a non-trivial zero of $\zeta(s)$ then so is $\bar{\rho}$. For $z \in \mathbb{H}$ one has

$$\varpi_n^{(\beta)}(z) = \lim_{m\to\infty} \overline{\sum_{\substack{\rho \\ 0<\operatorname{Im}\rho<T_m}} \frac{\overline{\sigma_{1-\rho/\beta}^{(\beta)}(n)}}{\zeta'(\rho)} e^{\rho z}}.$$

By using $\overline{\sigma_{1-\rho/\beta}^{(\beta)}(n)} = \sigma_{1-\bar{\rho}/\beta}^{(\beta)}(n)$ and $\zeta(\bar{s}) = \overline{\zeta(s)}$ we deduce that

$$\begin{aligned}\varpi_n^{(\beta)}(z) &= \overline{\sum_{\substack{\rho \\ 0<\operatorname{Im}\rho<T_m}} \frac{\overline{\sigma_{1-\rho/\beta}^{(\beta)}(n)}}{\zeta'(\rho)} e^{\rho z}} = \sum_{\substack{\rho \\ 0<\operatorname{Im}\rho<T_m}} \overline{\frac{\sigma_{1-\bar{\rho}/\beta}^{(\beta)}(n)}{\zeta'(\bar{\rho})} e^{\overline{\rho z}}} \\ &= \overline{\sum_{\substack{\rho \\ -T_m<\operatorname{Im}\rho<0}} \frac{\sigma_{1-\rho/\beta}^{(\beta)}(n)}{\zeta'(\rho)} e^{\rho \bar{z}}} = \overline{\hat{\varpi}_n^{(\beta)}(\bar{z})}.\end{aligned}$$

Invoking (4.27) with $z \in \mathbb{H}$, we see that

$$\begin{aligned}\varpi_n^{(\beta)}(z) &= \overline{\hat{\varpi}_n^{(\beta)}(\bar{z})} \\ &= -\overline{\varpi_n^{(\beta)}(\bar{z})} - \overline{\sum_{q=1}^{\infty} \frac{\mu(q)}{q} \sum_{k=0}^{\infty} \frac{1}{k!} \left\{ \left(e^{-\bar{z}} \frac{2\pi i}{q}\right)^k + \left(-e^{-\bar{z}} \frac{2\pi i}{q}\right)^k \right\} \sigma_{1+k/\beta}^{(\beta)}(n)} \\ &= -\overline{\varpi_n^{(\beta)}(\bar{z})} - \sum_{q=1}^{\infty} \frac{\mu(q)}{q} \sum_{k=0}^{\infty} \frac{1}{k!} \left\{ \left(e^{-z} \frac{2\pi i}{q}\right)^k + \left(-e^{-z} \frac{2\pi i}{q}\right)^k \right\} \sigma_{1+k/\beta}^{(\beta)}(n)\end{aligned}$$

and by complex conjugation for $z \in \hat{\mathbb{H}}$, and by analytic continuation for z with $y = \operatorname{Im}(z) = 0$. This proves the functional equation (4.10).

Another expression can be found which depends on the values of the Riemann zeta-function at odd integers. To that end,

$$\begin{aligned}A_n^{(\beta)}(z) &= -\sum_{q=1}^{\infty} \frac{\mu(q)}{q} \left(\sum_{k=0}^{\infty} \frac{1}{k!} \left(\left(e^{-z} \frac{2\pi i}{q}\right)^k + \left(-e^{-z} \frac{2\pi i}{q}\right)^k \right) \sigma_{1+k/\beta}^{(\beta)}(n) - 2\sigma_1^{(\beta)}(n) \right) \\ &= -\sum_{q=1}^{\infty} \frac{\mu(q)}{q} \sum_{k=1}^{\infty} \frac{1}{k!} \left(\left(e^{-z} \frac{2\pi i}{q}\right)^k + \left(-e^{-z} \frac{2\pi i}{q}\right)^k \right) \sigma_{1+k/\beta}^{(\beta)}(n) \\ &= -\sum_{k=1}^{\infty} \frac{1}{k!} (e^{-z} 2\pi i)^k \sigma_{1+k/\beta}^{(\beta)}(n) \sum_{q=1}^{\infty} \frac{\mu(q)}{q^{1+k}} \\ &\quad - \sum_{k=1}^{\infty} \frac{1}{k!} (-e^{-z} 2\pi i)^k \sigma_{1+k/\beta}^{(\beta)}(n) \sum_{q=1}^{\infty} \frac{\mu(q)}{q^{1+k}} \\ &= -\sum_{k=1}^{\infty} \frac{1}{k!} ((e^{-z} 2\pi i)^k + (-e^{-z} 2\pi i)^k) \sigma_{1+k/\beta}^{(\beta)}(n) \frac{1}{\zeta(1+k)} \\ &= -2 \sum_{k=1}^{\infty} \frac{(-1)^k (2\pi)^{2k}}{(2k)!} \frac{e^{-2kz} \sigma_{1+2k/\beta}^{(\beta)}(n)}{\zeta(2k+1)},\end{aligned}$$

since the even indexed terms vanish. Finally, if $z = x + iy$, then we are left with

$$|A_n^{(\beta)}(z)| \leq 2n^{\beta+1} \sum_{k=1}^{\infty} \frac{(2\pi n e^{-x})^{2k}}{(2k)!},$$

which converges absolutely. Thus $A_n^{(\beta)}(z)$ defines an entire function. This proves Theorem 4.1.9.

Finally, Theorem 4.1.10 follows easily from Theorem 4.1.9.

Chapter 5

On a class of functions that satisfies certain explicit formulae

5.1 Introduction and results

This chapter is taken from the Ramanujan Journal paper [KRR14], together with my two collaborators N. Robles and A. Roy.

5.1.1 Motivation for studying the Möbius function

The Möbius function μ is defined as

$$\mu(n) = \begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } p^2 | n \text{ for some prime } p, \\ (-1)^k, & \text{if } n \text{ is a product of } k \text{ distinct primes.} \end{cases} \tag{5.1}$$

If x denotes a positive real number, then the Mertens function M is defined by

$$M(x) = \sum_{n \leqslant x} \mu(n).$$

The interest in studying $\mu(n)$ and $M(x)$ comes from their connection to the distribution of the prime numbers. For instance (see §1.1 of [HL18]), the prime number theorem is equivalent to each of the statements

$$M(x) = o(x), \quad \sum_{n=1}^{\infty} \frac{\mu(n)}{n} = 0. \tag{5.2}$$

It is known that if $Q(x)$ denotes the number of positive squarefree numbers less than or equal to x then the asymptotics of $Q(x)$ are given by

$$Q(x) = \frac{x}{\zeta(2)} + O(\sqrt{x}).$$

The Möbius function highlights numbers which are not squarefree numbers. Denote by $d(n)$ the number of divisors of n, including 1 and itself. Since for any $\delta > 0$,

$$\lim_{n \to \infty} \frac{d(n)}{n^{\delta}} = 0,$$

the set of all positive integers divisible by only a bounded number, say k, of primes is equal to zero. Thus, in this sense, most integers $1 \leq n \leq N$ are divisible by a substantial number of primes. By studying the Mertens functions we are recording $+1$ if this large number is even and -1 if it is odd. Under this light then, it is suggestive to interpret this parity as being random, not unlike the flipping of a fair coin, e.g. $+1$ for heads and -1 for tails.

If we follow this probabilistic interpretation, then the prime number theorem in Mertens form (5.2) seems to indicate that if we toss a large number x of coins then number of heads minus the number of tails should be small when compared to the total number of coin tosses. The following quotation is from [IK04].

MÖBIUS RANDOMNESS LAW. The Möbius function $\mu(n)$ changes sign randomly so that for any "reasonable" sequence of numbers $\mathcal{A} = (a_n)$, the twisted sum

$$M(\mathcal{A}, x) = \sum_{n \leq x} \mu(n) a_n$$

is relatively small due to cancellation of its terms. On the other hand, the theory of

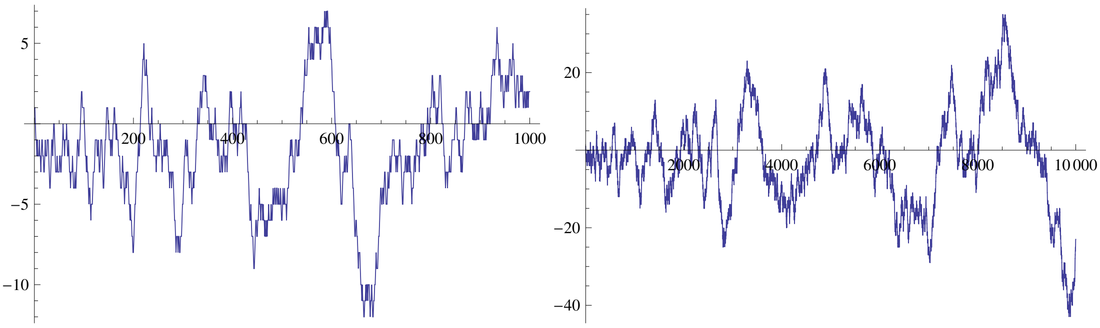

FIGURE 5.1: Plot of $M(x)$ for $1 \leq x \leq 1000$ (left) and $1 \leq x \leq 10000$ (right).

probability tells us that the aggregate number of heads minus the aggregate number of tails should be of the the order of the square root of the total number of toin cosses. This brings us to the following question. What is the upper order of $M(x)$? A trivial bound is given by $M(x) \leq x$, but this is not satisfactory. In the 1880's Mertens [Edw74] conjectured the following.

> *One has*
> $$M(x) \leq \sqrt{x}$$
> *for all sufficiently large x.*

Later in 1885, Stieltjes [Edw74] claimed a proof of this conjecture; however, this proof has never been published nor found amongst Stieltjes papers. It wasn't until 100 years later than te Riele and Odlyzko [OR85] disproved the Mertens' conjecture. Specifically they showed the following.

> *There are explicit constants $C_1 > 1$ and $C_2 < -1$ such that*
> $$\limsup_{x \to \infty} \frac{M(x)}{\sqrt{x}} \geq C_1, \quad \liminf_{x \to \infty} \frac{M(x)}{\sqrt{x}} \leq C_2.$$

This means that each of the inequalities $-\sqrt{x} \leq M(x)$ and $M(x) \leq \sqrt{x}$ fails for infinitely many x, or, equivalenly, $M(x) = \Omega_{\pm}(\sqrt{x})$. The proof of te Riele and Odlyzko does not provide a specific value of x for which $M(x) \geq \sqrt{x}$, but it is known that there is such an x for $x < 10^{156}$.

In [BT12], Best ad Trudigan gave an alternative disproof of the Mertens' conjecture and they showed that C_1 can be taken to be 1.6383 and C_2 to be -1.6383. The best unconditional estimate on the Mertens' function is (see Ivić [Ivi85, §12, p. 309])

$$M(x) \ll x \exp\left(-c_1 \log^{\frac{3}{5}} x (\log\log x)^{-\frac{1}{5}}\right),$$

where $c_1 > 0$; and the bound on the assumption of the Riemann hypothesis is (see Titchmarsh [Tit86, §14.26])

$$M(x) \ll x^{1/2} \exp\left(\frac{c_2 \log x}{\log\log x}\right),$$

for $c_2 > 0$. The best unconditional Ω-result for the Mertens function is

$$M(x) = \Omega_{\pm}\left(x^{\frac{1}{2}}\right),$$

and if $\zeta(s)$ has a zero of multiplicity m with $m > 1$, then

$$M(x) = \Omega_{\pm}\left(x^{\frac{1}{2}} (\log x)^{m-1}\right).$$

On the other hand, if the Riemann hypothesis is false, then

$$M(x) = \Omega_{\pm}\left(x^{\theta-\delta}\right),$$

where $\theta = \sup\limits_{\rho, \zeta(\rho)=0} \mathrm{Re}(\rho)$ and δ is any positive constant (see Ingham [Ing85, p. 82]).

5.1.2 Explicit formulae

An explicit formula is an equation which encapsulates certain arithmetical information and which involves the non-trivial zeros ρ of the Riemann zeta-function. Possibly the most famous one is due to Riemann (1859) (see e.g. Chapters 1 and 3 of [Edw74]) and von Mangoldt (1895). It is used to prove the prime number theorem. The von Mangoldt function is defined by

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^m \text{ for some } m \in \mathbb{N} \text{ and prime } p, \\ 0, & \text{otherwise.} \end{cases}$$

The sum of these is termed the Chebyshev function,

$$\psi(x) = \sum_{n \leqslant x} \Lambda(n).$$

We define $\psi_0(x)$ to be $\psi(x)$ except when $x = p^k$ in which case it takes the value halfway between the values to the left and to the right. If $x > 1$ then the explicit formula is

$$\psi_0(x) = x - \sum_{\rho} \frac{x^\rho}{\rho} - \log(2\pi) - \frac{1}{2}\log(1 - x^{-2}). \tag{5.3}$$

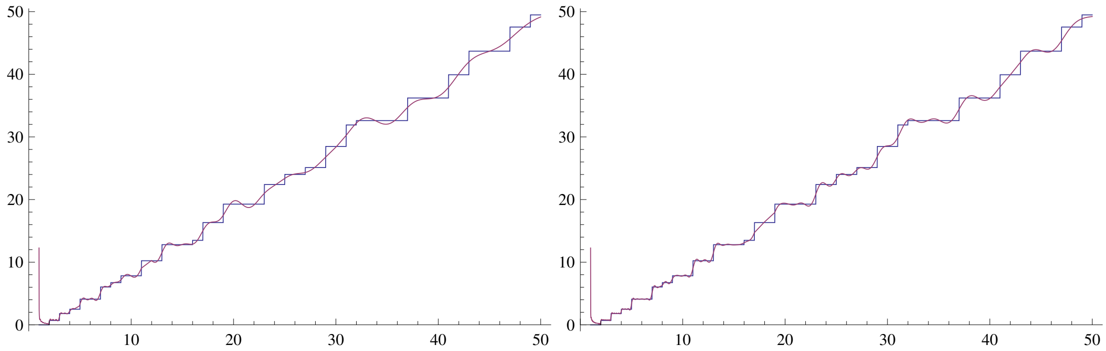

FIGURE 5.2: Plot of (5.3) for $1 \le x \le 50$ with 10 pairs of zeros (left) and 25 pairs of zeros (right).

5.1.2.1 Ramanujan's explicit formula

In 1918 Hardy and Littlewood (see §2.5 of [HL18] and §9.8 of [Tit86]) published an explicit formula suggested to them by Ramanujan. Under the benign assumption that the non-trivial zeros ρ are all simple, their explicit formula can be stated as follows.

Let a and b be two positive real numbers such that $ab = \pi$. Let φ and ψ be a pair of suitable cosine reciprocal functions[1]*. Let $Z_1(s)$ and $Z_2(s)$ be the Mellin transforms of $\varphi(s)$ and $\psi(s)$ respectively. Then*

$$\sqrt{a}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\varphi\left(\frac{a}{n}\right)-\sqrt{b}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\psi\left(\frac{b}{n}\right)=\frac{1}{2\sqrt{a}}\sum_{\rho}a^{\rho}\frac{Z_1(1-\rho)}{\zeta'(\rho)}$$
$$=-\frac{1}{2\sqrt{b}}\sum_{\rho}b^{\rho}\frac{Z_2(1-\rho)}{\zeta'(\rho)}, \tag{5.4}$$

provided the series involving ρ are convergent.

If we take $\varphi(x) = \psi(x) = \exp(-x^2)$, then it is easily seen that these functions are cosine reciprocal functions and that

$$Z_1(s) = Z_2(s) = \frac{1}{2}\Gamma\left(\frac{s}{2}\right).$$

In this case (5.4) becomes

$$\sqrt{a}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-a^2/n^2}-\sqrt{b}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-b^2/n^2}=\frac{1}{2\sqrt{a}}\sum_{\rho}a^{\rho}\frac{\Gamma(\frac{1-\rho}{2})}{\zeta'(\rho)}=-\frac{1}{2\sqrt{b}}\sum_{\rho}b^{\rho}\frac{\Gamma(\frac{1-\rho}{2})}{\zeta'(\rho)}, \tag{5.5}$$

provided, once again, that the series

$$\sum_{\rho}\alpha^{\rho}\frac{\Gamma(\frac{1-\rho}{2})}{\zeta'(\rho)}$$

[1]Two functions $f(x)$ and $g(x)$ are cosine reciprocal if

$$\frac{\sqrt{\pi}}{2}f(x)=\int_0^{\infty}g(u)\cos(2ux)du, \quad \frac{\sqrt{\pi}}{2}g(x)=\int_0^{\infty}f(u)\cos(2ux)du.$$

is convergent for $\alpha > 1$. Hardy and Littlewood credit Ramanujan for first providing (5.5) and later on for suggesting the generalization (5.4). They do not, however, state the conditions that φ and ψ must satisfy for (5.4) to hold. The arithmetical information is contained on the left-hand side of (5.4) and (5.5) and the analytic information is encoded in the sums involving the zeros on the right-hand side.

In 2013 Dixit [Dix13] gave a one variable generalization of (5.5). He showed the following result.

If we let a and b be positive reals such that $ab = \pi$ and $z \in \mathbb{C}$, then

$$\begin{aligned}\sqrt{a}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-a^2/n^2}\cosh\left(\frac{z}{n}\right) - \sqrt{b}e^{-z^2/4}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-a^2/n^2}\cos\left(\frac{z}{n}\right) \\ = \frac{1}{2\sqrt{a}}\sum_{\rho}a^{\rho}\frac{\Gamma(\frac{1-\rho}{2})}{\zeta'(\rho)}{}_1F_1\left(\frac{s}{2},\frac{1}{2},\frac{z^2}{4}\right) \\ = -\frac{1}{2\sqrt{b}}\sum_{\rho}b^{\rho}\frac{\Gamma(\frac{1-\rho}{2})}{\zeta'(\rho)}{}_1F_1\left(\frac{s}{2},\frac{1}{2},-\frac{z^2}{4}\right),\end{aligned} \tag{5.6}$$

provided the series involving ρ are convergent and where ${}_1F_1$ denotes the confluent hypergeometric function.

Clearly, if $z = 0$ then (5.6) reduces to (5.5).

In [Dix12], Dixit obtained a character analogue of (5.5). To state his result we recall the following notation of the theory of Dirichlet L-functions. Suppose that χ is a character mod q. The indicator function $\mathfrak{h}$ is defined by

$$\mathfrak{h} = \begin{cases} 0 & \text{if } \chi \text{ is even,} \\ 1 & \text{if } \chi \text{ is odd.} \end{cases}$$

The Gauss sum $\tau(\chi)$ is defined by

$$\tau(\chi) = \sum_{m=1}^{q}\chi(m)e^{2\pi i m/q}.$$

With this in mind, Dixit's second result is as follows.

Let a and b be two positive real numbers such that $ab = \pi$ and let χ denote a primitve Dirichlet character $\bmod q$ such that $\chi(-1) = (-1)^{\mathfrak{h}}$. If the non-trivial zeros ρ of $L(s,\chi)$ are all simple then one has

$$\begin{aligned}a^{\mathfrak{h}+1/2}\sqrt{\tau(\chi)}\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n^{1+\mathfrak{h}}}e^{-a^2/(qn^2)} - b^{\mathfrak{h}+1/2}\sqrt{\tau(\bar{\chi})}\sum_{n=1}^{\infty}\frac{\bar{\chi}(n)\mu(n)}{n^{1+\mathfrak{h}}}e^{-b^2/(qn^2)} \\ = q\frac{\sqrt{\tau(\chi)}}{2\sqrt{a}}\sum_{\rho}\left(\frac{b}{q^{1/2}}\right)^{\rho}\frac{\Gamma(\frac{1+\mathfrak{h}-\rho}{2})}{L'(\rho,\chi)} \\ = -q\frac{\sqrt{\tau(\bar{\chi})}}{2\sqrt{b}}\sum_{\rho}\left(\frac{b}{q^{1/2}}\right)^{\rho}\frac{\Gamma(\frac{1+\mathfrak{h}-\rho}{2})}{L'(\rho,\bar{\chi})}\end{aligned} \tag{5.7}$$

provided the series involving ρ are convergent.

Later in [DRZ16] Roy, Dixit and Zaharescu found the character analogue of (5.6) and in [ARZ15] a generalization of (5.6) to Hecke forms.

The transformations in (5.4), (5.5), (5.6) and (5.7) exhibit a transformation of the type $x \to 1/x$, which is an analogue of the Poisson summation formula. These kinds of transformation formula have broad interest in different branches of mathematics. In this article they established a class of reciprocal functions, as well as a class of arithmetical functions obtained from a reduced Selberg class, which satisfies the transformation formula mentioned above. At the end of the introduction they provided examples where they obtained the above transformations as special cases. Furthermore, they obtained some new transformations that were not in the literature.

Let us suppose that $A_1 > 0$ and $T > 0$. We define the bracketing condition $\mathcal{B}$ on a sum involving the zeros $\rho = \beta + i\gamma$ and $\rho' = \beta' + i\gamma'$ of $\zeta(s)$ to be a summation where the terms are bracketed in such a way that two terms for which

$$\left|\gamma - \gamma'\right| < \exp(-A_1|\gamma|/\log|\gamma|) + \exp(-A_1|\gamma'|/\log|\gamma'|) \tag{5.8}$$

are included in the same bracket. When a sum over ρ satisfies the bracketing condition $\mathcal{B}$ we will write $\sum_{\rho\in\mathcal{B}} f(\rho)$.

We define the bracketing condition $\mathcal{B}_\chi$ on a sum involving the zeros $\rho = \beta + i\gamma$ and $\rho' = \beta' + i\gamma'$ of $L(s,\chi)$ to be a summation where the terms are bracketed in such a way that two terms for which

$$\left|\gamma - \gamma'\right| < \exp(-A_1|\gamma|/\log|\gamma| + 3) + \exp(-A_1|\gamma'|/\log|\gamma'| + 3) \tag{5.9}$$

are included in the same bracket. Similarly, when a sum over ρ satisfies the bracketing condition $\mathcal{B}_\chi$ we will write $\sum_{\rho\in\mathcal{B}_\chi} f(\rho)$. If we assume that the zeros of $\zeta(s)$ satisfy the bracketing condition $\mathcal{B}$ then one can drop the assumption of convergence of the series on the right hand sides of (5.4), (5.5) and (5.6); the same happens for (5.7), if we assume the zeros of $L(s,\chi)$ satisfy the bracketing condition $\mathcal{B}_\chi$.

The size and the distribution of such bracketings are unknown but their existence is widely accepted, see §2.5 of [HL18] and §9.8 of [Tit86].

In fact, it is expected the pairs of zeros $\{\rho, \rho'\}$ that need to be bracketed together in Ramanujan's explicit formula to occur very rarely. For results on the correlation of zeros of L-functions, the reader is referred to Montgomery [Mon73], Rudnick and Sarnak [RS96], Katz and Sarnak [KS99b; KS99a], Murty and Perelli [MP99], and Murty and Zaharescu [MZ02].

5.1.2.2 Titchmarsh explicit formula

An explicit formula for the Mertens function was first published in 1951 by Titchmarsh on the assumption of the Riemann hypothesis (see §14.27 of [Tit86]). Specifically,

On RH and the simplicity of the non-trivial zeros, there exists a sequence T_ν, $\nu \le T_\nu \le \nu + 1$, such that

$$M(x) = -2 + \lim_{\nu\to\infty} \sum_{|\nu|<T_\nu} \frac{x^\rho}{\rho\zeta'(\rho)} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(2\pi/x)^{2n}}{(2n)!n\zeta(2n+1)} \tag{5.10}$$

if x is not an integer. If x is an integer, $M(x)$ is to be replaced by

$$M(x) - \tfrac{1}{2}\mu(x).$$

Note that, unlike RH, the assumption that the zeros are all simple is made for convenience. Indeed, this condition can be relaxed and zeros with higher multiplicity can be accommodated at the cost of making the explicit formula much more complicated. Since it is widely believed that all zeros of the zeta function are simple we shall operate under this assumption throughout.
In 1991, Bartz (see [Bar91a] and [Bar91b]) proved (5.10) unconditionally. The plot of the explicit formula is shown below.

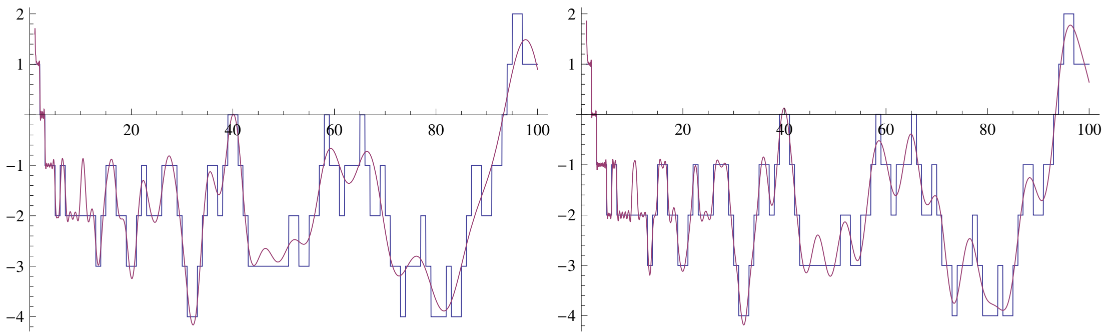

FIGURE 5.3: Plot of (5.10) for $1 \leq x \leq 100$ with 10 pairs of zeros (left) and 25 pairs of zeros (right).

5.1.2.3 Weil explicit formula

In 1952, Weil (see §5.5 of [IK04] and [Wei52]) published a different kind of explicit formula for the von Mangoldt function.

> *Suppose that f is $\mathcal{C}^\infty$ and compactly supported. Moreover, denote by F its Mellin transform. Then*

$$\sum_{\rho} F(\rho) + \sum_{n=1}^{\infty} F(-2n) = F(1) + \sum_{n=1}^{\infty} \Lambda(n) f(n).$$

Note that this is a different version of the formula appearing in Chapter 3, because it involves the Mellin transform.

In order to state the main theorems, we first need to introduce some further concepts.

5.1.3 Hankel transformations

Two functions $\varphi(x)$ and $\psi(x)$ are said to be reciprocal under the Hankel transformation of order ν if

$$\varphi(x) = \int_0^\infty (ux)^{\frac{1}{2}} J_\nu(ux)\psi(u)\, du \quad \text{and} \quad \psi(x) = \int_0^\infty (ux)^{\frac{1}{2}} J_\nu(ux)\varphi(u)\, du, \tag{5.11}$$

where $J_\nu(x)$ is the Bessel function of the first kind of order ν defined by

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{\nu+2n}}{n!\Gamma(\nu+n+1)}.$$

The existence of such reciprocity was first shown by Titchmarsh, see [Tit22] and [Tit48]. In particular he showed the following.

If $\varphi(s)$ is integrable in the sense of Lebesgue and $\nu \geq -\frac{1}{2}$ then

$$\int_0^a (ux)^{\frac{1}{2}} J_\nu(ux)\varphi(u)\, du$$

converges in mean to a function $\psi(x)$ of integrable square in $(0, \infty)$ as $a \to \infty$.

Hankel transformations reduce to Fourier's cosine and sine transforms for $\nu = -\frac{1}{2}$ and $\nu = \frac{1}{2}$ respectively. The Mellin transforms of $\varphi(x)$ and $\psi(x)$ are defined, as usual, by

$$Z_1(s) = \int_0^\infty x^{s-1}\varphi(x)dx, \quad Z_2(s) = \int_0^\infty x^{s-1}\psi(x)dx. \tag{5.12}$$

Their inverse Mellin transforms are given by

$$\varphi(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} Z_1(s)x^{-s}ds, \quad \psi(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} Z_2(s)x^{-s}ds. \tag{5.13}$$

The value of c will depend on the nature of the functions φ and ψ.

Definition 5.1.1. *Let $0 < \omega \leq \pi$ and $\alpha < \frac{1}{2}$. If $f(z)$ is such that*

i) $f(z)$ is analytic of $z = re^{i\theta}$ regular in the wedge domain defined by $|\theta| < \omega$,

ii) $f(z)$ satisfies the bounds

$$f(z) = \begin{cases} O(|z|^{-\alpha-\epsilon}) & \text{if } |z| \text{ is small,} \\ O(e^{-|z|}) & \text{if } |z| \text{ is large,} \end{cases} \tag{5.14}$$

for every positive ϵ and uniformly in any angle $|\theta| < \omega$,

then we say that f belongs to the class K and write $f(z) \in K(\omega, \alpha)$.

5.1.4 Main results

Equipped with these notions our first result is as follows.

Theorem 5.1.1. *Suppose that F is an element of the Selberg class defined in Chapter 1, §1.2.1 with $d_F = 1$. Let $\nu \geq -\frac{1}{2}$ and the first H-invariant (1.8) to be $H_F(1) = -\nu - \frac{1}{2}$. Let $\frac{\pi}{4} < \omega \leq \pi$, $\alpha < \frac{1}{2}$ and $\varphi, \psi \in K(\omega, \alpha)$ be reciprocal functions under the Hankel transformation of order ν. Let $Z_1(s)$ and $Z_2(s)$ be defined as above and let x be a positive real numbers. If a and b are two positive real numbers such that $ab = 2\pi$, then there exists a sequence $\{T_l\}$ of positive numbers that satisfies the following.*

i) If $q_F = 1$ then

$$\sum_{n=1}^{\infty} \mu(n)\varphi\left(\frac{n}{x}\right) = \lim_{l\to\infty} \sum_{-T_l < \mathrm{Im}(\rho) < T_l} \frac{Z_1(\rho)}{\zeta'(\rho)}x^\rho + \sqrt{2\pi}\sum_{k=1}^{\infty} \frac{(-1)^k Z_2(1+k)}{(k!)^2\zeta(1+k)}\left(\frac{x}{2\pi}\right)^{-k}. \tag{5.15}$$

ii) If $q_F \geq 2$ then there exists a primitive Dirichlet character $\chi \mod q_F$ with $\chi(-1) = -2\nu$ such that

$$\sum_{n=1}^{\infty} \mu(n)\chi(n)\varphi\left(\frac{n}{x}\right) = \lim_{l\to\infty} \sum_{-T_l < \mathrm{Im}(\rho) < T_l} \frac{Z_1(\rho)}{L'(\rho, \chi)}x^\rho +$$

$$+ i^{\frac{1}{2}+\nu}\frac{\sqrt{2\pi}}{\tau(\chi)}\sum_{k=0}^{\infty}\frac{(-1)^k Z_2(1+k)}{(k!)^2 L(1+k,\bar{\chi})}\left(\frac{qx}{2\pi}\right)^{-k} + \frac{Z_1(s_0)}{L'(s_0,\chi)}x^{s_0} \tag{5.16}$$

on the assumption that the Riemann hypothesis for Dirichlet L-functions is true, and where s_0 denotes a hypothetical Landau-Siegel zero.

Equation (5.15) is reminiscent of the Weil explicit formula except that $\Lambda(n)$ is replaced by $\mu(n)$. Similar formulae due to Berndt [Ber71] and Ferrar (see [Fer35], [Fer37], and §2.9 of [Tit48]) for the divisor function $d(n)$ exist as well. Extensions of the Weil explicit formula (5.15) to generalized von Mangoldt functions and other arithmetical functions such as the Liouville λ function can be found in [MRR].

If, for example, one takes the step function

$$\varphi(x) = \begin{cases} 1, & \text{if } x < 1, \\ \frac{1}{2}, & \text{if } x = 1, \\ 0, & \text{if } x > 1, \end{cases}$$

then its cosine reciprocal would be given by

$$\psi(x) = \frac{2}{\sqrt{\pi}}\int_0^1 \cos(2ux)du = \frac{2\cos x}{\sqrt{\pi}}\frac{\sin x}{x}.$$

The Mellin transforms of φ and ψ would then become

$$Z_1(s) = \frac{1}{s}, \quad (\mathrm{Re}(s) > 0) \quad Z_2(s) = -\frac{2^{1-s}}{\sqrt{\pi}}\cos\left(\frac{\pi s}{2}\right)\Gamma(s-1) \quad (0 < \mathrm{Re}(s) < 2),$$

and otherwise by analytic continuation. In this case (5.15) would reduce to Titchmarsh's explicit formula (5.10).

The second result is as follows.

Theorem 5.1.2. *Suppose that F is an element of the Selberg class with $d_F = 1$. Let $\nu \geq -\frac{1}{2}$ and $H_F(1) = -\nu - \frac{1}{2}$. Let $\frac{\pi}{4} < \omega \leq \pi$, $\alpha < \frac{1}{2}$ and $\varphi, \psi \in K(\omega, \alpha)$ be reciprocal functions under the Hankel transformation of order ν. Let $Z_1(s)$ and $Z_2(s)$ be defined as above. If a and b are two positive real numbers such that $ab = 2\pi$, then one has the following.*

i) If $q_F = 1$, then

$$\sqrt{a}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\varphi\left(\frac{a}{n}\right) - \sqrt{b}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\psi\left(\frac{b}{n}\right) = \frac{1}{\sqrt{a}}\sum_{\rho\in\mathcal{B}}a^{\rho}\frac{Z_1(1-\rho)}{\zeta'(\rho)}$$
$$= -\frac{1}{\sqrt{b}}\sum_{\rho\in\mathcal{B}}b^{\rho}\frac{Z_2(1-\rho)}{\zeta'(\rho)}. \tag{5.17}$$

ii) If $q_F \geq 2$, then there exists a primitive Dirichlet character χ mod q_F with $\chi(-1) = -2\nu$ such that

$$\sqrt{a}\sqrt{\tau(\chi)}\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n}\varphi\left(\frac{a}{q^{1/2}n}\right) - \sqrt{b}\sqrt{\tau(\bar{\chi})}\sum_{n=1}^{\infty}\frac{\bar{\chi}(n)\mu(n)}{n}\psi\left(\frac{b}{q^{1/2}n}\right) \tag{5.18}$$

$$
\begin{aligned}
&= \frac{q^{1/2}\sqrt{\tau(\chi)}}{a^{1/2}} \sum_{\rho\in\mathcal{B}_\chi} \left(\frac{a}{q^{1/2}}\right)^{\rho} \frac{Z_1(1-\rho)}{L'(\rho,\chi)} \\
&= -\frac{q^{1/2}\sqrt{\tau(\bar{\chi})}}{b^{1/2}} \sum_{\rho\in\mathcal{B}_\chi} \left(\frac{b}{q^{1/2}}\right)^{\rho} \frac{Z_2(1-\rho)}{L'(\rho,\bar{\chi})}.
\end{aligned}
$$

One can see that the condition $H_F(1) = \nu - \frac{1}{2}$ is necessary. This condition naturally leads us to make the following conjecture.

Conjecture 5.1.1. *Let F be an element in the Selberg class with $d_F = 1$. Let $\nu \geq -\frac{1}{2}$, $\frac{\pi}{2} < \omega \leq \pi$ and $\varphi, \psi \in K(\omega,\alpha)$ be reciprocal under the Hankel transformation of order ν. Then (5.17) holds only when $\nu = -1/2$ and (5.18) holds only when $\nu = \pm 1/2$.*

Remark 5.1.1. *The following special cases are to be noted.*

1. *Let $\varphi(x) = \psi(x) = x^{(\nu+1/2)}e^{-\frac{x^2}{2}}$ for $\nu = \pm 1/2$. Clearly $\varphi, \psi \in K(\omega,\alpha)$. Also*

$$
Z_1(s) = Z_2(s) = \left(\frac{1}{2}\right)^{\left(\frac{\nu}{2}-\frac{3}{4}\right)} 2^{\frac{s}{2}}\Gamma\left(\frac{s+\nu+1/2}{2}\right).
$$

If we substitute the above values of φ, ψ, Z_1 and Z_2 in (5.18) then we obtain (5.7).

2. *Let $\varphi(x) = e^{-x^2-z^2/2}\cosh(zx)$ and $\psi(x) = e^{-x^2+z^2/2}\cos(zx)$. One can see that $\varphi, \psi \in K(\omega,\alpha)$ and they are reciprocal under cosine transformations i.e., $\nu = -1/2$. Their Mellin transformations are given by*

$$
\begin{aligned}
Z_1(s) &= \frac{1}{2}e^{-\frac{z^2}{8}}\Gamma\left(\frac{s}{2}\right){}_1F_1\left(\frac{s}{2},\frac{1}{2};\frac{z^2}{4}\right), \\
Z_2(s) &= \frac{1}{2}e^{\frac{z^2}{8}}\Gamma\left(\frac{s}{2}\right){}_1F_1\left(\frac{s}{2},\frac{1}{2};-\frac{z^2}{4}\right).
\end{aligned}
$$

If we substitute the above values of φ, ψ, Z_1 and Z_2 in (5.17) and (5.18) then we obtain (5.6) and [DRZ16, Theorem 1.2, part i)] respectively.

3. *Let $\varphi(x) = e^{-x^2-z^2/2}\sinh(zx)$ and $\psi(x) = e^{-x^2+z^2/2}\sin(zx)$. One can see that $\varphi, \psi \in K(\omega,\alpha)$ and that they are reciprocal under sine transformations, i.e. $\nu = 1/2$. Their Mellin transformations are given by*

$$
\begin{aligned}
\Phi(s) &= \frac{z}{2}e^{-\frac{z^2}{8}}\Gamma\left(\frac{1+s}{2}\right){}_1F_1\left(\frac{1+s}{2},\frac{3}{2};\frac{z^2}{4}\right) \\
Z_2(s) &= \frac{z}{2}e^{\frac{z^2}{8}}\Gamma\left(\frac{1+s}{2}\right){}_1F_1\left(\frac{1+s}{2},\frac{3}{2};-\frac{z^2}{4}\right).
\end{aligned}
$$

If we substitute the above values of φ, ψ, Φ and Z_2 in (5.18), then we obtain [DRZ16, Theorem 1.2, part ii)].

The following corollaries are new transformations in the literature. It is not difficult to find pairs of reciprocal functions and obtain new formulae from (5.17). For instance, one could take the pair of cosine reciprocal functions

$$
\varphi(x) = e^{-x}, \quad \psi(x) = \frac{2}{\sqrt{\pi}}\frac{1}{1+4x^2},
$$

with Mellin transforms

$$Z_1(s) = \Gamma(s), \quad Z_2(s) = 2^{-s}\sqrt{\pi}\csc\left(\frac{\pi s}{2}\right),$$

valid for $\mathrm{Re}(s) > 0$ and $0 < \mathrm{Re}(s) < 2$, respectively, and obtain the following.

Corollary 5.1.1.

$$\begin{aligned}&\sqrt{a}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}e^{-a/n} - 2\sqrt{\frac{b}{\pi}}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\frac{1}{1+4(\frac{b}{n})^2}\\ &= \frac{1}{a^{1/2}}\sum_{\rho\in B}a^{\rho}\frac{\Gamma(1-\rho)}{\zeta'(\rho)} = -\frac{1}{2}\sqrt{\frac{\pi}{b}}\sum_{\rho\in B}\frac{(2a)^{\rho}}{\zeta'(\rho)}\csc\left(\frac{\pi(1-\rho)}{2}\right),\end{aligned}$$

However, the symmetry is more striking on the left hand-side when we take a pair of *self-reciprocal* functions. For the following corollaries, a and b will denote two positive real numbers satisfying $ab = 2\pi$ and the non-trivial zeros of $\zeta(s)$ and $L(s,\chi)$ are all assumed to be simple. Here χ denotes the primitive Dirichlet character $\bmod q$.

Corollary 5.1.2. *Let χ be odd. Then we have*

$$\begin{aligned}&\sqrt{a\tau(\chi)}\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n}\left(\frac{1}{e^{a\sqrt{2\pi/qn}}-1} - \frac{n}{a}\sqrt{\frac{q}{2\pi}}\right)\\ &- \sqrt{b\tau(\bar{\chi})}\sum_{n=1}^{\infty}\frac{\bar{\chi}(n)\mu(n)}{n}\left(\frac{1}{e^{b\sqrt{2\pi/qn}}-1} - \frac{n}{b}\sqrt{\frac{q}{2\pi}}\right)\\ &= \sqrt{\frac{q\tau(\chi)}{2\pi a}}\sum_{\rho\in B_\chi}\left(\frac{(2\pi)^{1/2}a}{q^{1/2}}\right)^{\rho}\frac{\Gamma(1-\rho)\zeta(1-\rho)}{L'(\rho,\chi)}.\end{aligned} \tag{5.19}$$

Corollary 5.1.3. *Let χ be even. Then*

$$\begin{aligned}&\sqrt{a}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\operatorname{sech}\left(\sqrt{\frac{\pi}{2}}\frac{a}{n}\right) - \sqrt{b}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\operatorname{sech}\left(\sqrt{\frac{\pi}{2}}\frac{a}{n}\right)\\ &\qquad = \sqrt{\frac{1}{2\pi a}}\sum_{\rho\in B}(2^{\frac{3}{2}}\pi^{\frac{1}{2}}a)^{\rho}\frac{\Gamma(1-\rho)(\zeta(1-\rho,\frac{1}{4})-\zeta(1-\rho,\frac{3}{4}))}{\zeta'(\rho)}\end{aligned} \tag{5.20}$$

as well as

$$\begin{aligned}&\sqrt{a\tau(\chi)}\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n}\operatorname{sech}\left(\sqrt{\frac{\pi}{2q}}\frac{a}{n}\right) - \sqrt{b\tau(\bar{\chi})}\sum_{n=1}^{\infty}\frac{\bar{\chi}(n)\mu(n)}{n}\left(\sqrt{\frac{\pi}{2q}}\frac{b}{n}\right)\\ &= \sqrt{\frac{q\tau(\chi)}{2\pi a}}\sum_{\rho\in B_\chi}\left(\frac{2^{\frac{3}{2}}\pi^{\frac{1}{2}}a}{q^{1/2}}\right)^{\rho}\frac{\Gamma(1-\rho)(\zeta(1-\rho,\frac{1}{4})-\zeta(1-\rho,\frac{3}{4}))}{L'(\rho,\chi)},\end{aligned} \tag{5.21}$$

where $\zeta(s,\alpha)$ denotes the Hurwitz zeta-function.

Corollary 5.1.4. *Let $\chi(-1) = -2\nu$. Then we have*

$$a\sum_{n=1}^{\infty}\frac{\mu(n)}{n^{3/2}}J_{-\frac{1}{4}}\left(\frac{1}{2}\frac{a^2}{n^2}\right) - b\sum_{n=1}^{\infty}\frac{\mu(n)}{n^{3/2}}J_{-\frac{1}{4}}\left(\frac{1}{2}\frac{a^2}{n^2}\right) = \frac{1}{\sqrt{2a}}\sum_{\rho\in B}\left(\frac{a}{2}\right)^{\rho}\frac{1}{\zeta'(\rho)}\frac{\Gamma(\frac{1}{4}-\frac{\rho}{4})}{\Gamma(\frac{1}{2}+\frac{\rho}{4})}. \tag{5.22}$$

For $\nu = \pm 1/2$,

$$\frac{a\sqrt{\tau(\chi)}}{q^{1/4}}\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n^{3/2}}J_{\frac{\nu}{2}}\left(\frac{a^2}{2qn^2}\right) - \frac{b\sqrt{\tau(\bar{\chi})}}{q^{1/4}}\sum_{n=1}^{\infty}\frac{\bar{\chi}(n)\mu(n)}{n^{3/2}}J_{\frac{\nu}{2}}\left(\frac{b^2}{2qn^2}\right)$$
$$= \sqrt{\frac{q\tau(\chi)}{2a}}\sum_{\rho\in B_\chi}\left(\frac{a}{2q^{1/2}}\right)^{\rho}\frac{1}{L'(\rho,\chi)}\frac{\Gamma(\frac{1}{4}-\frac{\rho}{4})}{\Gamma(\frac{1}{2}+\frac{\rho}{4})}.$$

Corollary 5.1.5. *Let $K_\nu(x)$ be the modified Bessel function of the second kind of order ν. Let $\chi(-1) = -2\nu$. Then for $\mathrm{Re}(z) > 0$ and $\nu = \frac{1}{4}$ we have*

$$\begin{aligned}
&\sqrt{a}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\left(\frac{a^2}{n^2}+z^2\right)^{-\frac{1}{8}}K_{\frac{1}{4}}\left(z\sqrt{z^2+\frac{a^2}{n^2}}\right)\\
&-\sqrt{b}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\left(\frac{b^2}{n^2}+z^2\right)^{-\frac{1}{8}}K_{\frac{1}{4}}\left(z\sqrt{z^2+\frac{b^2}{n^2}}\right)\\
&=\frac{1}{\sqrt{2a}}\sum_{\rho\in B}\left(\frac{a}{2^{1/2}}\right)^{\rho}\frac{\Gamma(\frac{1-\rho}{2})K_{-\frac{1}{2}(\frac{1}{2}-\rho)}(z^2)}{\zeta'(\rho)},
\end{aligned} \tag{5.23}$$

and for $\mu = \pm 1/2$,

$$\begin{aligned}
&\frac{a^{1+\mu}\sqrt{\tau(\chi)}}{q^{\frac{1}{2}(\frac{1}{2}+\mu)}}\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n^{1+\frac{1}{2}+\mu}}\left(z^2+\frac{a^2}{qn^2}\right)^{\frac{1}{4}(-\mu-1)}K_{\frac{1}{2}(\mu+1)}\left(z\sqrt{z^2+\frac{a^2}{qn^2}}\right)\\
&-\frac{a^{1+\mu}\sqrt{\tau(\bar{\chi})}}{q^{\frac{1}{2}(\frac{1}{2}+\mu)}}\sum_{n=1}^{\infty}\frac{\bar{\chi}(n)\mu(n)}{n}\left(z^2+\frac{b^2}{qn^2}\right)^{\frac{1}{4}(-\mu-1)}K_{\frac{1}{2}(\mu+1)}\left(z\sqrt{z^2+\frac{b^2}{qn^2}}\right)\\
&=2^{\frac{2\mu-1}{4}}\frac{\sqrt{q\tau(\chi)}}{a}\sum_{\rho\in B_\chi}\left(\frac{a}{(2q)^{1/2}}\right)^{\rho}\frac{\Gamma(\frac{3}{4}+\frac{1}{2}\mu-\frac{1}{2}\rho)K_{-\frac{1}{2}(\frac{1}{2}-\rho)}(z^2)}{L'(\rho,\chi)}.
\end{aligned} \tag{5.24}$$

Let us recall that the Weber parabolic cylinder functions $D_n(x)$ are defined by [Mit38, pp. 205-206]:

$$D_n(x) = \frac{\Gamma(\frac{1}{2})2^{\frac{n}{2}}e^{-\frac{1}{4}x^2}}{\Gamma(\frac{1}{2}-\frac{n}{2})}{}_1F_1(-\tfrac{n}{2},\tfrac{1}{2};\tfrac{x^2}{2}) + \frac{\Gamma(-\frac{1}{2})2^{\frac{n}{2}-\frac{1}{2}}e^{-\frac{1}{4}x^2}}{\Gamma(-\frac{n}{2})}{}_1F_1(\tfrac{1}{2}-\tfrac{n}{2},\tfrac{3}{2};\tfrac{x^2}{2})$$

for all real numbers n and x.

Corollary 5.1.6. *One has*

$$\sqrt{a}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\exp\left(\frac{a^2}{4n^2}\right)D_{-2}\left(\frac{a}{n}\right) - \sqrt{b}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\exp\left(\frac{b^2}{4n^2}\right)D_{-2}\left(\frac{b}{n}\right)$$

$$= \frac{1}{2^{1/2}a^{1/2}} \sum_{\rho \in B} (2^{1/2}a)^{\rho} \frac{\Gamma(1-\rho)\Gamma(\frac{1}{2}+\frac{1}{2}\rho)}{\zeta'(\rho)}.$$

Moreover,

$$\begin{aligned}
&\sqrt{a\tau(\chi)} \sum_{n=1}^{\infty} \frac{\chi(n)\mu(n)}{n} \exp\left(\frac{a^2}{4qn^2}\right) D_{-2}\left(\frac{a}{q^{1/2}n}\right) \\
&- \sqrt{b\tau(\bar{\chi})} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)\mu(n)}{n} \exp\left(\frac{b^2}{4qn^2}\right) D_{-2}\left(\frac{b}{q^{1/2}n}\right) \\
&= \sqrt{\frac{q\tau(\chi)}{2a}} \sum_{\rho \in B_\chi} \left(\frac{2^{1/2}a}{q^{1/2}}\right)^{\rho} \frac{\Gamma(1-\rho)\Gamma(\frac{1}{2}+\frac{1}{2}\rho)}{L'(\rho,\chi)}.
\end{aligned}$$

Finally,

$$\begin{aligned}
&a\sqrt{\frac{a\tau(\chi)}{q}} \sum_{n=1}^{\infty} \frac{\chi(n)\mu(n)}{n^2} \exp\left(\frac{a^2}{4qn^2}\right) D_{-4}\left(\frac{a}{q^{1/2}n}\right) \\
&- b\sqrt{\frac{b\tau(\bar{\chi})}{q}} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)\mu(n)}{n^2} \exp\left(\frac{b^2}{4qn^2}\right) D_{-4}\left(\frac{b}{q^{1/2}n}\right) \\
&= \frac{1}{3}\sqrt{\frac{q\tau(\chi)}{a}} \sum_{\rho \in B_\chi} \left(\frac{a}{2^{3/2}q^{1/2}}\right)^{\rho} \frac{\Gamma(1-\frac{1}{2}\rho)\Gamma(1+\rho)\Gamma(\frac{3}{2}-\frac{1}{2}\rho)}{\Gamma(\frac{1}{2}+\frac{1}{2}\rho)L'(\rho,\chi)}.
\end{aligned}$$

Corollary 5.1.7. *For* $\chi(-1) = -1$*,*

$$\begin{aligned}
&\sqrt{a\tau(\chi)} \sum_{n=1}^{\infty} \frac{\chi(n)\mu(n)}{n} \exp\left(\frac{a^2}{4qn^2}\right) D_{-1}\left(\frac{a}{q^{1/2}n}\right) \\
&- \sqrt{b\tau(\bar{\chi})} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)\mu(n)}{n} \exp\left(\frac{b^2}{4qn^2}\right) D_{-1}\left(\frac{b}{q^{1/2}n}\right) \\
&= \frac{1}{2}\sqrt{\frac{q\tau(\chi)}{a}} \sum_{\rho \in B_\chi} \left(\frac{2^{1/2}a}{q^{1/2}}\right)^{\rho} \frac{\Gamma(1-\rho)\Gamma(\frac{1}{2}\rho)}{L'(\rho,\chi)}.
\end{aligned}$$

Straightforward computation shows that

$$\frac{\cosh\left(x\sqrt{\frac{\pi}{2}}\right)}{\cosh(x\sqrt{2\pi})} \quad \text{and} \quad \frac{1}{1+2\cosh\left(2x\sqrt{\frac{\pi}{3}}\right)} \tag{5.25}$$

are self-reciprocal Hankel transformations of order $\nu = -1/2$ and

$$\frac{\sinh\left(x\sqrt{\frac{\pi}{2}}\right)}{\cosh(x\sqrt{2\pi})} \quad \text{and} \quad \frac{\sinh\left(x\sqrt{\frac{\pi}{3}}\right)}{2\cosh\left(2x\sqrt{\frac{\pi}{3}}\right)-1} \tag{5.26}$$

are self-reciprocal Hankel transformations of order $\nu = 1/2$. In a similar fashion to the above corollaries one can obtain transformation formulas for the functions (5.25) and (5.26). There exist many self-reciprocal Hankel transformations of order $\nu = \pm\frac{1}{2}$ in the literature and a transformation formula can be obtained from each one of them. The functions mentioned in the above corollaries are well known in the literature and have many applications in, e.g., mathematical physics, see e.g. [WW62, pp. 235-579].

5.2 Preliminary Lemmas

We will use the following lemmas to prove our main theorems.

Lemma 5.2.1. *Let $\varphi, \psi \in L^2(0,\infty)$ be two reciprocal Hankel transform of order ν. Then*

$$\varphi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2^{\frac{it}{2}}\Gamma\left(\frac{\nu}{2}+\frac{1}{2}+\frac{it}{2}\right)\Phi\left(\frac{1}{2}+it\right)x^{-\frac{1}{2}-it}\,dt, \tag{5.27}$$

$$\psi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2^{\frac{it}{2}}\Gamma\left(\frac{\nu}{2}+\frac{1}{2}+\frac{it}{2}\right)\Psi\left(\frac{1}{2}+it\right)x^{-\frac{1}{2}-it}\,dt, \tag{5.28}$$

for two functions Φ, Ψ such that the integrals are mean square integralble, $2^{\frac{it}{2}}\Gamma\left(\frac{\nu}{2}+\frac{1}{2}+\frac{it}{2}\right)\cdot \Phi\left(\frac{1}{2}+it\right)$ and $2^{\frac{it}{2}}\Gamma\left(\frac{\nu}{2}+\frac{1}{2}+\frac{it}{2}\right)\cdot\Psi\left(\frac{1}{2}+it\right)$ belong to $L^2(-\infty,\infty)$, and

$$\Phi\left(\frac{1}{2}-it\right) = \Psi\left(\frac{1}{2}+it\right). \tag{5.29}$$

Proof. Supoose that φ belongs to $L^2(0,\infty)$. One can see that

$$\int_0^\infty \varphi^2(x)\,dx = \int_{-\infty}^\infty \varphi^2(e^x)e^x\,dx.$$

Hence $F(x) := \varphi(e^x)e^{x/2} \in L^2(-\infty,\infty)$. Then from the theory of Fourier transforms (see [Tit48, Chapter 3, p. 69]) it follows that

$$Z_1\left(\frac{1}{2}+it\right) = \int_{-\infty}^{\infty} F(x)e^{itx}\,dx = \int_0^\infty \varphi(x)x^{-\frac{1}{2}+it}\,dx \tag{5.30}$$

exists as a mean square integral for almost all t. Also $Z_1\left(\frac{1}{2}+it\right) \in L^2(-\infty,\infty)$ and

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Z_1\left(\frac{1}{2}+it\right)e^{-ixt}\,dt. \tag{5.31}$$

The above integral is also a mean square integral. In other words, (5.31) can be written as

$$\varphi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Z_1\left(\frac{1}{2}+it\right)x^{-\frac{1}{2}+it}\,dt. \tag{5.32}$$

Similarly we obtain

$$\psi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Z_2\left(\frac{1}{2}+it\right)x^{-\frac{1}{2}+it}\,dt. \tag{5.33}$$

Let us consider two functions Φ and Ψ such that

$$Z_1\left(\frac{1}{2}+it\right) = 2^{\frac{it}{2}}\Gamma\left(\frac{\nu}{2}+\frac{1}{2}+\frac{it}{2}\right)\Phi\left(\frac{1}{2}+it\right) \tag{5.34}$$

and

$$Z_2\left(\frac{1}{2}+it\right) = 2^{\frac{it}{2}}\Gamma\left(\frac{\nu}{2}+\frac{1}{2}+\frac{it}{2}\right)\Psi\left(\frac{1}{2}+it\right). \tag{5.35}$$

Inserting the above equalities in (5.32) and (5.33) we obtain (5.27) and (5.28). Now we complete the proof by proving (5.29). For all $n \geq -1/2, y > 0$ and $x > 0$ we have

$$\int_0^y \sqrt{ux} J_\nu(ux)\, du = \frac{y(xy)^{\nu+\frac{1}{2}} {}_1F_2\left({\frac{\nu}{2}+\frac{3}{4} \atop \frac{\nu}{2}+\frac{7}{4}, \nu+1}; -\frac{x^2y^2}{4}\right)}{2^\nu(\nu+3/2)\Gamma(\nu+1)}. \tag{5.36}$$

The right-hand side of (5.36) belongs to $L^2(0,\infty)$ and the Mellin transform is given by

$$\int_0^\infty \frac{y(xy)^{\nu+\frac{1}{2}} {}_1F_2\left({\frac{\nu}{2}+\frac{3}{4} \atop \frac{\nu}{2}+\frac{7}{4}, \nu+1}; -\frac{x^2y^2}{4}\right)}{2^\nu(\nu+3/2)\Gamma(\nu+1)} x^{-\frac{1}{2}+it}\, dt = \frac{2^{it} y^{\frac{1}{2}-it} \Gamma\left(\frac{\nu}{2}+\frac{1}{2}+\frac{it}{2}\right)}{\left(\frac{1}{2}-it\right)\Gamma\left(\frac{\nu}{2}+\frac{1}{2}-\frac{it}{2}\right)}. \tag{5.37}$$

We also have that $\varphi \in L^2(0,\infty)$ and its Mellin transform is given by (5.30). Hence by an analogue of Plancherel's theorem for Mellin transform (see [Tit48, Theorem 72]) we have

$$\begin{aligned}
&\int_0^\infty \varphi(x) \frac{y(xy)^{\nu+\frac{1}{2}} {}_1F_2\left({\frac{\nu}{2}+\frac{3}{4} \atop \frac{\nu}{2}+\frac{7}{4}, \nu+1}; -\frac{x^2y^2}{4}\right)}{2^\nu(\nu+3/2)\Gamma(\nu+1)}\, dx \qquad (5.38)\\
&= \frac{1}{2\pi}\int_{-\infty}^\infty \frac{2^{it} y^{\frac{1}{2}-it} \Gamma\left(\frac{\nu}{2}+\frac{1}{2}+\frac{it}{2}\right)}{\left(\frac{1}{2}-it\right)\Gamma\left(\frac{\nu}{2}+\frac{1}{2}-\frac{it}{2}\right)} Z_1\left(\frac{1}{2}-it\right) dt\\
&= \frac{1}{2\pi}\int_{-\infty}^\infty 2^{\frac{it}{2}} \Gamma\left(\frac{\nu}{2}+\frac{1}{2}+\frac{it}{2}\right) \Phi\left(\frac{1}{2}-it\right) \frac{y^{\frac{1}{2}-it}}{\frac{1}{2}-it}\, dt,
\end{aligned}$$

in the ultimate step we have used (5.34). Now from (5.11) we have

$$\psi(u) = \lim_{a\to\infty} \int_0^a \sqrt{ux} J_\nu(ux)\varphi(x)\, dx,$$

where the limit converges in the sense of mean-square. Therefore for all $x > 0, y > 0$ and $\nu \geq -1/2$ we find that

$$\begin{aligned}
\int_0^y \psi(u)\, du &= \lim_{a\to\infty} \int_0^y \int_0^a \sqrt{ux} J_\nu(ux)\varphi(x)\, dx\, du \qquad (5.39)\\
&= \lim_{a\to\infty} \int_0^a \varphi(x) \frac{y(xy)^{\nu+\frac{1}{2}} {}_1F_2\left({\frac{\nu}{2}+\frac{3}{4} \atop \frac{\nu}{2}+\frac{7}{4}, \nu+1}; -\frac{x^2y^2}{4}\right)}{2^\nu(\nu+3/2)\Gamma(\nu+1)}\, dx\\
&= \int_0^\infty \varphi(x) \frac{y(xy)^{\nu+\frac{1}{2}} {}_1F_2\left({\frac{\nu}{2}+\frac{3}{4} \atop \frac{\nu}{2}+\frac{7}{4}, \nu+1}; -\frac{x^2y^2}{4}\right)}{2^\nu(\nu+3/2)\Gamma(\nu+1)}\, dx.
\end{aligned}$$

The left hand side of (5.39) is

$$\begin{aligned}
\int_0^y \psi(u)\, du &= \frac{1}{2\pi}\int_0^y \int_{-\infty}^\infty 2^{\frac{it}{2}} \Gamma\left(\frac{\nu}{2}+\frac{1}{2}+\frac{it}{2}\right) \Psi\left(\frac{1}{2}+it\right) u^{-\frac{1}{2}-it}\, dt\, du \qquad (5.40)\\
&= \frac{1}{2\pi}\left[\lim_{X\to\infty}\int_0^y\int_0^X + \lim_{Y\to\infty}\int_0^y\int_{-Y}^0\right] 2^{\frac{it}{2}} \Gamma\left(\frac{\nu}{2}+\frac{1}{2}+\frac{it}{2}\right)\\
&\quad \times \Psi\left(\frac{1}{2}+it\right) u^{-\frac{1}{2}-it}\, dt\, du \qquad (5.41)
\end{aligned}$$

$$= \frac{1}{2\pi} \int_{-\infty}^{\infty} 2^{\frac{it}{2}} \Gamma\left(\frac{\nu}{2} + \frac{1}{2} + \frac{it}{2}\right) \Psi\left(\frac{1}{2} + it\right) \frac{y^{\frac{1}{2}-it}}{\frac{1}{2} - it}\, dt.$$

By (5.39) we see the right-hand sides of (5.38) and (5.40) are equal. Hence from [Tit48, Theorem 32] we conclude that

$$\Phi\left(\frac{1}{2} - it\right) = \Psi\left(\frac{1}{2} + it\right).$$

□

Lemma 5.2.2. *Let φ and ψ be reciprocal functions under the Hankel transformation of order ν defined in (5.11). Let $\varphi, \psi \in K(\omega, \alpha)$. Then there exist two regular functions Φ and Ψ such that*

$$\varphi(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{\frac{s}{2}-\frac{1}{4}} \Gamma\left(\frac{s}{2} + \frac{\nu}{2} + \frac{1}{4}\right) \Phi(s) x^{-s}\, ds, \tag{5.42}$$

$$\psi(x) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} 2^{\frac{s}{2}-\frac{1}{4}} \Gamma\left(\frac{s}{2} + \frac{\nu}{2} + \frac{1}{4}\right) \Psi(s) x^{-s}\, ds \tag{5.43}$$

for $c > 0$. Moreover Φ and Ψ satisfy the following:

1. *$\Phi(s) = \Psi(1-s)$ for all $s \in \mathbb{C}$,*
2. *$\Psi(s) = O(e^{(\frac{\pi}{4}-\omega+\epsilon)|t|})$ for every positive ϵ and uniformly for $\sigma \in \mathbb{R}$.*

Proof. Since $\varphi, \psi \in K(\omega, \alpha)$, the right-hand sides of (5.12) are absolutely convergent. Then it follows that $Z_1(s)$ and $Z_2(s)$ are regular in $\alpha < \sigma$. Let

$$Z_1(s) = 2^{\frac{s}{2}-\frac{1}{4}} \Gamma\left(\frac{s}{2} + \frac{\nu}{2} + \frac{1}{4}\right) \Phi(s), \tag{5.44}$$

and

$$Z_2(s) = 2^{\frac{s}{2}-\frac{1}{4}} \Gamma\left(\frac{s}{2} + \frac{\nu}{2} + \frac{1}{4}\right) \Psi(s). \tag{5.45}$$

Hence by (5.34) and (5.35) of Lemma 5.2.1, we deduce that $\Phi(s)$ and $\Psi(s)$ also regular in this region. One can see $\varphi, \psi \in L^2$. Therefore from (5.29) of Lemma 5.2.1, $\Psi(s) = \Phi(1-s)$ for $\sigma = 1/2$. Thus, by analytic continuation $\Psi(s) = \Phi(1-s)$ for $\alpha < \sigma$ and hence for all $s \in \mathbb{C}$. Also (5.42) and (5.43) hold for $\alpha < c = \sigma$. Let us consider the line along any radius vector r and angle θ, where $|\theta| < \omega$. Then by Cauchy's theorem we can deform the integral (5.12) to

$$Z_1(\sigma + it) = \int_0^{\infty} r^{\sigma+it-1} e^{i\theta(\sigma+it)} \varphi(re^{i\theta})\, dr,$$

where $\theta, t > 0$. Therefore

$$\begin{aligned} |Z_1(\sigma + it)| &= e^{-\theta t} \left| \int_0^{\infty} r^{\sigma+it-1} e^{i\theta(\sigma+it-1)} \varphi(re^{\theta})\, dr \right| \\ &\le e^{-\theta t} \int_0^{\infty} r^{\sigma-1} |\varphi(re^{\theta})| dr = O(e^{-|\theta||t|}), \end{aligned} \tag{5.46}$$

since $\varphi \in K(\omega, \alpha)$. Now combining (5.44), (5.46) and Stirling's formula (A.12), we obtain

$$\Psi(1-s) = \Phi(s) = O(e^{(\frac{\pi}{4}-|\theta|)|t|}) = O(e^{(\frac{\pi}{4}-\omega+\eta)|t|}), \tag{5.47}$$

for every positive η. This proves the lemma. □

The following results due to Montgomery, [Mon77]; Ramachandra and Balasubramanian, [Ram74], [Ram77] and [BR77] will enable us to prove Theorem 5.1.1 with $d_F = q_F = 1$ without the assumption of the Riemann Hypothesis.

Lemma 5.2.3. *For any given $\varepsilon > 0$ there exists a $T_0 = T_0(\varepsilon)$ such that for $T \geq T_0$ the following holds: there exists a $t \in [T, 2T]$ for which*

$$|\zeta(\sigma \pm it)|^{-1} < c_1 t^{\varepsilon}$$

uniformly for $-1 \leq \sigma \leq 2$ with an absolute constant $c_1 > 0$.

For the case where $q_F > 1$, the analogue results are due to Soundararajan, [Sou08]; Lamzouri, [Lam11]. However, this latter depends on the truth of the Riemann hypothesis for Dirichlet L-functions.

Lemma 5.2.4. *Assume the Generalized Riemann Hypothesis for Dirichlet L-functions. For any given $\varepsilon > 0$ and primitive Dirichlet character $\chi \bmod q$ there exists a $T_0 = T_0(\varepsilon, q)$ such that $T \geq T_0$ the following holds: there exists a $t \in [T, 2T]$ for which*

$$|L(\sigma \pm it, \chi)|^{-1} < c(q) t^{\varepsilon}$$

uniformly for $-1 \leq \sigma \leq 2$ with an absolute constant $c(q) > 0$.

An intermediate result we will be using is due to Ahlgren, Berndt, Yee and Zaharescu [ABY+02].

Lemma 5.2.5. *If χ is a primitive character of conductor N and k is an integer ≥ 2 such that $\chi(-1) = (-1)^k$ then one has*

$$\frac{(k-2)!N^{k-2}\tau(\chi)}{2^{k-1}\pi^{k-2}i^{k-2}} L(k-1, \bar{\chi}) = L'(2-k, \chi). \tag{5.48}$$

5.3 Proof of Theorem 5.1.1

i) Let F be a Selberg L-function of degree $d_F = 1$ and conductor $q_F = 1$. Then by Lemma 1.2.1 we see that $F(s) = \zeta(s)$, where $\zeta(s)$ is the Riemann zeta-function. Therefore there is only one gamma factor in the completed Selberg L-function of F for which $r_j = 0$ and $\lambda_j = 1/2$. From (1.8) we see that $H_F(1) = -1$ when $r_j = 0$ and hence $\nu = 1/2$. Therefore $\varphi, \psi \in K(\omega, \alpha)$ is a pair of reciprocal sine transformations. Now

$$\begin{aligned}\sum_{n=1}^{\infty} \mu(n)\varphi\left(\frac{n}{x}\right) &= \frac{1}{2\pi i}\sum_{n=1}^{\infty}\mu(n)\int_{\lambda-i\infty}^{\lambda+i\infty} Z_1(s)\left(\frac{x}{n}\right)^s ds \\ &= \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} Z_1(s)x^s\left(\sum_{n=1}^{\infty}\frac{\mu(n)}{n^s}\right) ds. \end{aligned} \tag{5.49}$$

By Lemma 5.2.2 $Z_1(s) = O(e^{(-\omega+\eta)|t|})$ for every positive η. For $1 < \lambda < 2$ the sum inside the above integral is absolute convergent. Therefore the far right-hand side of the above equalities is absolutely convergent, which justifies the interchange of the summation and integration. Recall the Dirichlet series valid for $\mathrm{Re}(s) > 1$ of the Möbius function

$$\sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} = \frac{1}{\zeta(s)}.$$

From (5.34) we find that the simple poles of $Z_1(s)$ are at $s = -2k+1$ for $k = 0, 1, 2, \cdots$. For $1 < \lambda < 2$ and $-1 < c < 0$ we consider the positively oriented closed contour $\Omega = [c - iT, c + iT, \lambda + iT, \lambda - iT]$ where $T > 0$. Therefore by the residue theorem

$$\frac{1}{2\pi i} \int_{\Omega} \frac{Z_1(s)}{\zeta(s)} x^s \, ds = \sum_{-T<\mathrm{Im}(\rho)<T} \lim_{s\to\rho} (s-\rho) \frac{Z_1(s)}{\zeta(s)} x^s = \sum_{-T<\mathrm{Im}(\rho)<T} \frac{Z_1(\rho)}{\zeta'(\rho)} x^{\rho}. \tag{5.50}$$

The functional equation of $\zeta(s)$ is given by

$$\zeta(s) = \pi^{s-\frac{1}{2}} \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)} \zeta(1-s). \tag{5.51}$$

From Lemma 5.2.2 we have

$$Z_1(s) = 2^{s-\frac{1}{2}} \frac{\Gamma\left(\frac{1+s}{2}\right)}{\Gamma(1-\frac{s}{2})} Z_2(1-s). \tag{5.52}$$

Hence by using (5.51), (5.52) and the duplication formula of gamma function (A.6), we find that

$$\int_{c-iT}^{c+iT} \frac{Z_1(s)}{\zeta(s)} x^s \, ds = \sqrt{2\pi} \int_{c-iT}^{c+iT} \left(\frac{x}{2\pi}\right)^s \frac{\Gamma(s)}{\Gamma(1-s)} \frac{Z_2(1-s)}{\zeta(1-s)} \, ds. \tag{5.53}$$

Now we consider the positively oriented contour Ω' with sides $[-N - \frac{1}{2} - iT, c - iT]$, $[c - iT, c + iT]$, $[c + iT, -N - \frac{1}{2} + iT]$ and $[-N - \frac{1}{2} + iT, -N - \frac{1}{2} - iT]$. The poles of the integrand of the right-hand side integral of (5.53) are at $k = -1, -2, -3, \ldots$. By the residue theorem we have

$$\frac{\sqrt{2\pi}}{2\pi i} \int_{\Omega'} \left(\frac{x}{2\pi}\right)^s \frac{\Gamma(s)}{\Gamma(1-s)} \frac{Z_2(1-s)}{\zeta(1-s)} \, ds = \sqrt{2\pi} \sum_{k=1}^{N} \frac{(-1)^k Z_2(1+k)}{(k!)^2 \zeta(1+k)} \left(\frac{x}{2\pi}\right)^{-k}. \tag{5.54}$$

Therefore by Lemma 5.2.2 and using Stirling's formula (A.13), we have

$$\begin{aligned} &\int_{-N-\frac{1}{2}-iT}^{-N-\frac{1}{2}+iT} \left(\frac{x}{2\pi}\right)^s \frac{\Gamma(s)}{\Gamma(1-s)} \frac{Z_2(1-s)}{\zeta(1-s)} ds \ll \\ &\int_{-T}^{T} \left(\frac{x}{2\pi}\right)^{-N-\frac{1}{2}} \frac{e^{2(N+1)-2(N+1)\log(\sqrt{t^2+(N+1/2)^2})}}{e^{(\pi+\omega+\eta)|t|}} dt, \end{aligned} \tag{5.55}$$

which tends to zero as $N \to \infty$ for any fixed T. Combining (5.54) and (5.55) we find that

$$\sqrt{2\pi} \int_{c-iT}^{c+iT} \left(\frac{x}{2\pi}\right)^s \frac{\Gamma(s)}{\Gamma(1-s)} \frac{Z_2(1-s)}{\zeta(1-s)} \, ds$$

$$= \sqrt{2\pi}\sum_{k=1}^{\infty}\frac{(-1)^k Z_2(1+k)}{(k!)^2\zeta(1+k)}\left(\frac{x}{2\pi}\right)^{-k} \tag{5.56}$$
$$+\sqrt{2\pi}\left[\int_{-\infty-iT}^{c-iT}+\int_{-\infty+iT}^{c+iT}\right]\left(\frac{x}{2\pi}\right)^{s}\frac{\Gamma(s)}{\Gamma(1-s)}\frac{Z_2(1-s)}{\zeta(1-s)}\,ds.$$

Similarly as with (5.55) we have

$$\int_{-\infty\pm iT}^{c\pm iT}\left(\frac{x}{2\pi}\right)^{s}\frac{\Gamma(s)}{\Gamma(1-s)}\frac{Z_2(1-s)}{\zeta(1-s)}\,ds = O\left(\int_{-\infty}^{c}\left(\frac{x}{2\pi}\right)^{\sigma}\frac{e^{1-2\sigma+(2\sigma-1)\log(\sqrt{T^2+\sigma^2})}}{e^{(\pi+\omega+\eta)T}}\,d\sigma\right)$$
$$= O\left(\frac{1}{e^{(\pi+\omega+\eta)T}}\right). \tag{5.57}$$

Now by Lemmas 5.2.2 and 5.2.3 we have

$$\int_{c\pm iT}^{\lambda\pm iT}\frac{Z_1(s)}{\zeta(s)}x^s\,ds = O(T^{\epsilon}e^{(-\omega+\eta)T}), \tag{5.58}$$

where T runs through a sequence $\{T_l\}$ with $T_l > T_0(\epsilon)$ and $T_l \to \infty$. Here ϵ and η are any positive numbers. Now combine (5.49), (5.50), (5.53), (5.54), (5.55) and (5.58) to conclude that

$$\sum_{n=1}^{\infty}\mu(n)\varphi\left(\frac{n}{x}\right) = \lim_{l\to\infty}\sum_{-T_l<\operatorname{Im}\rho<T_l}\frac{Z_1(\rho)}{\zeta'(\rho)}x^{\rho} + \sqrt{2\pi}\sum_{k=1}^{\infty}\frac{(-1)^k Z_2(1+k)}{(k!)^2\zeta(1+k)}\left(\frac{x}{2\pi}\right)^{-k}.$$

This proves part i) of Theorem 5.1.1.

ii) In this case we consider that F is an L-function of degree $d_F = 1$ and conductor $q_F \geq 2$. Using Lemma 1.2.1 we find that $F(s) = L(s,\chi)$ for some Dirichlet primitive character $\bmod\ q_F$. Therefore the completed L-function of F contains only one gamma factor and hence $r_j = 0$ or $r_j = 1/2$. Since ν is real then $\operatorname{Im}(H_F(1)) = 0$ and hence $H_F(1) = -1$ or $H_F(1) = 0$. By Lemma 5.2.2 we know that $\Phi(s)$ is analytic on the whole complex plane. Therefore the poles of $Z_1(s)$ are at the poles of $\Gamma\left(\frac{s}{2}+\frac{\nu}{2}+\frac{1}{4}\right)$. If $\nu = -1/2$ then $s = 0$ is a pole of $Z_1(s)$. For the sake of brevity we will prove the case where χ is an even character $\bmod\ q_F$; that is, when $\nu = 1/2$. The other case is handled in a similar fashion. In this case $Z_1(s)$ is analytic for $\operatorname{Re}(s) > -1$. Arguing as in part i), we have

$$\sum_{n=1}^{\infty}\mu(n)\chi(n)\varphi\left(\frac{n}{x}\right) = \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty}\frac{Z_1(s)}{L(s,\chi)}x^s\,ds. \tag{5.59}$$

Consider the positively oriented contour Ω mentioned in part i). By the residue theorem one can find that

$$\frac{1}{2\pi i}\oint_{\Omega}\frac{Z_1(s)}{L(s,\chi)}x^s\,ds = \frac{Z_1(0)}{L'(0,\chi)} + \sum_{-T<\operatorname{Im}(\rho)<T}\frac{Z_1(\rho)}{L'(\rho,\chi)}x^{\rho}, \tag{5.60}$$

where the ρ's denote the non-trivial zeros $L(s,\chi)$, assumed to be simple for notational convenience. If there is a Landau-Siegel zero (see §14 of [Dav66]) at $s = s_0$ then we would have to add the extra term

$$\operatorname*{Res}_{s=s_0}\frac{Z_1(s)}{L(s,\chi)}x^s = \frac{Z_1(s_0)}{L'(s_0,\chi)}x^{s_0}.$$

We note that this hypothetical zero is real and simple. Moreover, in [Chu05] it was proved that for a condutor q up to 200000 there are no Landau-Siegel zeros. Using the functional equation of Lemma 5.2.2 and the relation in Lemma 5.2.5, we find that

$$\frac{Z_1(0)}{L'(0,\chi)} = \frac{\sqrt{2\pi}}{\tau(\chi)}\frac{Z_2(1)}{L(1,\bar{\chi})}. \tag{5.61}$$

Proceeding as in the proof of part i), we have

$$\begin{aligned}\int_{c-iT}^{c+iT} \frac{Z_1(s)}{L(s,\chi)} x^s\, ds &= \frac{\sqrt{2\pi}}{\tau(\chi)} \int_{c-iT}^{c+iT} \left(\frac{qx}{2\pi}\right)^s \frac{\Gamma(s)}{\Gamma(1-s)} \frac{Z_2(1-s)}{L(1-s,\bar{\chi})}\, ds \\ &= \frac{\sqrt{2\pi}}{\tau(\chi)} \sum_{k=1}^{\infty} \frac{(-1)^k Z_2(1+k)}{(k!)^2 L(1+k,\bar{\chi})} \left(\frac{qx}{2\pi}\right)^{-k} \\ &+ \frac{\sqrt{2\pi}}{\tau(\chi)} \left[\int_{-\infty-iT}^{c-iT} + \int_{-\infty+iT}^{c+iT}\right] \left(\frac{qx}{2\pi}\right)^s \frac{\Gamma(s)}{\Gamma(1-s)} \frac{Z_2(1-s)}{L(1-s,\bar{\chi})}\, ds.\end{aligned} \tag{5.62}$$

Using Lemma 5.2.2 and (A.13) we obtain the bounds for $\int_{-\infty-iT}^{c-iT}$ and $\int_{-\infty+iT}^{c+iT}$ of the form (5.57). Using Lemmas 5.2.2 and 5.2.4 we obtain a bound for the horizontal integral of (5.60) which is of the form (5.58). Combining (5.59), (5.60), (5.61) and (5.62) we conclude the proof.

5.4 Proof of Theorem 5.1.2 and Corollaries

Proof of Theorem 5.1.2. i) By repeating a similar argument as in the previous proof we deduce that if $d_F = q_F = 1$, then $F(s) = \zeta(s)$. This case is already sketched in [HL18, p. 160] and the missing ingredient comes from the definition of the K class which allows us to get rid of the far left and horizontal integrals in the path of integration shown in Figure 5.4.

ii) In this case we consider F to be a Selberg L-function of degree $d_F = 1$ and conductor $q_F \geq 2$. Using Lemma 1.2.1 we find that $F(s) = L(s,\chi)$ for some Dirichlet primitive character $\bmod q_F$. Therefore the completed L-function of F contains only one gamma factor and hence $r_j = 0$ or $1/2$. Since ν is real we have $\mathrm{Im}(H_F(1)) = 0$ and hence $H_F(1) = -1$ or $H_F(1) = 0$.

Suppose $H_F = -1$, then $\nu = -1/2$ and χ is an even primitive Dirichlet character $\bmod q_F$. Therefore $\varphi, \psi \in K(\omega, \alpha)$ is a pair of cosine reciprocal functions. For $1 < \lambda < 1+\delta$ and $-1 < c < 0$ we consider the positively oriented closed contour $\Omega = [c-iT, c+iT, \lambda+iT, \lambda-iT]$ where $T > 0$. Recall that the functions Z_1 and Z_2 both have simple poles at $s = 0$. Hence from (5.34) and (5.35) we find that Φ and Ψ are analytic and non zero at $s = 0$. Furthermore, by the residue theorem,

$$\frac{1}{2\pi i}\oint_\Omega x^{-s} Z_1(s) ds = \operatorname*{Res}_{s=0} x^{-s} Z_k(s) = 2^{3/4}\Phi(0),$$

and

$$\frac{1}{2\pi i}\oint_\Omega x^{-s} Z_2(s) ds = \operatorname*{Res}_{s=0} x^{-s} Z_k(s) = 2^{3/4}\Psi(0).$$

By the use of the bound in Lemma 5.2.2 and Stirling's formula (A.13), the integrals along the horizontal lines of the contour Ω tend to zero as $T \to \infty$. Since (5.42) and

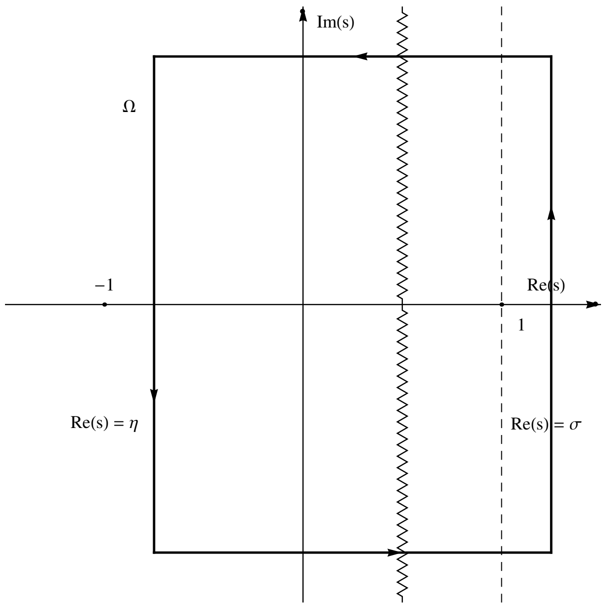

FIGURE 5.4: The rectangular contour Ω.

(5.43) hold for $\lambda > 1$ we have the following cases

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} x^{-s} Z_k(s)ds = \begin{cases} \varphi(x) - 2^{3/4}\Phi(0), & \text{if } k=1, \\ \psi(x) - 2^{3/4}\Psi(0), & \text{if } k=2. \end{cases} \tag{5.63}$$

If χ is an even primitive character of modulus q then $L(s,\chi)$ satisfies the functional equation

$$\frac{1}{L(1-s,\chi)} = \frac{\tau(\bar{\chi})}{q^{1/2}}\left(\frac{q}{\pi}\right)^{1/2-s}\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})}\frac{1}{L(s,\bar{\chi})},$$

for all complex values s. If we use the fact that $ab = 2\pi$ and couple this equation with (5.34), (5.35) and the functional equation of Φ and Ψ in Lemma 5.2.2, then we obtain

$$\frac{1}{2\pi i}\oint_{\Omega}\left(\frac{a}{q^{1/2}}\right)^{-s}\frac{Z_1(s)}{L(1-s,\chi)}ds = \frac{1}{2\pi i}\oint_{\Omega}\frac{\tau(\bar{\chi})}{(2\pi)^{1/2}}\left(\frac{b}{q^{1/2}}\right)^{s}\frac{Z_2(1-s)}{L(s,\bar{\chi})}ds. \tag{5.64}$$

By absolute convergence, with $c = \mathrm{Re}(s) < 0$, we may write

$$\begin{aligned}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{a}{q^{1/2}}\right)^{-s}\frac{Z_1(s)}{L(1-s,\chi)}ds &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{a}{q^{1/2}}\right)^{-s}\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n^{1-s}}Z_1(s)ds \\ &= \sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{a}{q^{1/2}n}\right)^{-s}Z_1(s)ds \\ &= \sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n}\varphi\left(\frac{a}{q^{1/2}n}\right) - \frac{2^{3/4}\Phi(0)}{L(1,\chi)},\end{aligned}$$

where we have used the case $k = 1$ of (5.63). Similarly, with $\lambda = \mathrm{Re}(s) > 1$, we have

$$\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty}\frac{\tau(\bar{\chi})}{(2\pi)^{1/2}}\left(\frac{b}{q^{1/2}}\right)^{s}\frac{Z_2(1-s)}{L(s,\bar{\chi})}ds$$

$$
\begin{aligned}
&= \frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty} \frac{\tau(\bar{\chi})b^s}{(2\pi)^{1/2}q^{s/2}} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)\mu(n)}{n^s} Z_2(1-s)ds \\
&= \frac{\tau(\bar{\chi})b}{(2\pi)^{1/2}q^{1/2}} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)\mu(n)}{n} \frac{1}{2\pi i}\int_{1-\lambda-i\infty}^{1-\lambda+i\infty} \left(\frac{b}{q^{1/2}n}\right)^{-w} Z_2(w)dw \\
&= \frac{\tau(\bar{\chi})b}{(2\pi)^{1/2}q^{1/2}} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)\mu(n)}{n} \psi\left(\frac{b}{q^{1/2}n}\right) - \frac{\tau(\bar{\chi})b}{(2\pi)^{1/2}q^{1/2}} \frac{2^{3/4}\Psi(0)}{L(1,\bar{\chi})}
\end{aligned}
$$

by making the change $w = 1-s$ and using the case $k = 2$ of (5.63). Now, we may use either side of (5.64) to evaluate the residues:

- for the non-trivial zeros ρ of $L(s,\chi)$, which we assume are all simple, we have

$$
\sum_{\rho} \operatorname*{Res}_{s=\rho} \frac{\tau(\bar{\chi})}{(2\pi)^{1/2}} \left(\frac{b}{q^{1/2}}\right)^s \frac{Z_2(1-s)}{L(s,\bar{\chi})} = \frac{\tau(\bar{\chi})}{(2\pi)^{1/2}} \sum_{\rho} \left(\frac{b}{q^{1/2}}\right)^{\rho} \frac{Z_2(1-\rho)}{L'(\rho,\bar{\chi})};
$$

- at $s = 1$ we have a simple pole coming from the $Z_2(1-s)$ function

$$
\operatorname*{Res}_{s=1} \frac{\tau(\bar{\chi})}{(2\pi)^{1/2}} \left(\frac{b}{q^{1/2}}\right)^s \frac{Z_2(1-s)}{L(s,\bar{\chi})} = -\frac{\tau(\bar{\chi})}{(2\pi)^{1/2}} \frac{b}{q^{1/2}} \frac{2^{3/4}\Psi(0)}{L(1,\bar{\chi})};
$$

- at $s = 0$ we have a trivial and simple zero of $L(s,\bar{\chi})$ and we know that $Z_2(1-s)$ is analytic and non zero, so

$$
\operatorname*{Res}_{s=0} \frac{\tau(\bar{\chi})}{(2\pi)^{1/2}} \left(\frac{b}{q^{1/2}}\right)^s \frac{Z_2(1-s)}{L(s,\bar{\chi})} = \frac{\tau(\bar{\chi})}{(2\pi)^{1/2}} \frac{Z_2(1)}{L'(0,\bar{\chi})} = \frac{2^{3/4}\Phi(0)}{L(1,\chi)},
$$

where we have used Lemma 5.2.5 with $N = q$ and $k = 2$ in the last equality. Consequently, by the residue theorem we have

$$
\begin{aligned}
\frac{\tau(\bar{\chi})b}{(2\pi)^{1/2}q^{1/2}} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)\mu(n)}{n} \psi\left(\frac{b}{q^{1/2}n}\right) - \sum_{n=1}^{\infty} \frac{\chi(n)\mu(n)}{n} \varphi\left(\frac{a}{q^{1/2}n}\right) \\
= \frac{\tau(\bar{\chi})}{(2\pi)^{1/2}} \sum_{\rho} \left(\frac{b}{q^{1/2}}\right)^{\rho} \frac{Z_2(1-\rho)}{L'(\rho,\bar{\chi})}.
\end{aligned}
$$

Multiplying both sides by $-\sqrt{a}\sqrt{\tau(\chi)}$ and using the fact that $\sqrt{\tau(\chi)\tau(\bar{\chi})} = q^{1/2}$ we have the desired result for even characters

$$
\begin{aligned}
&\sqrt{a}\sqrt{\tau(\chi)} \sum_{n=1}^{\infty} \frac{\chi(n)\mu(n)}{n} \varphi\left(\frac{a}{q^{1/2}n}\right) - \sqrt{b}\sqrt{\tau(\bar{\chi})} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)\mu(n)}{n} \psi\left(\frac{b}{q^{1/2}n}\right) \\
&= -q^{1/2} \frac{\sqrt{\tau(\bar{\chi})}}{b^{1/2}} \sum_{\rho} \left(\frac{b}{q^{1/2}}\right)^{\rho} \frac{Z_2(1-\rho)}{L'(\rho,\bar{\chi})}. \qquad (5.65)
\end{aligned}
$$

We note that if we had used the other side of (5.64) instead, then the result would have been

$$
\sqrt{a}\sqrt{\tau(\chi)} \sum_{n=1}^{\infty} \frac{\chi(n)\mu(n)}{n} \varphi\left(\frac{a}{q^{1/2}n}\right) - \sqrt{b}\sqrt{\tau(\bar{\chi})} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)\mu(n)}{n} \psi\left(\frac{b}{q^{1/2}n}\right)
$$

$$= q^{1/2}\frac{\sqrt{\tau(\chi)}}{a^{1/2}}\sum_{\rho}\left(\frac{a}{q^{1/2}}\right)^{\rho}\frac{Z_1(1-\rho)}{L'(\rho,\chi)}. \tag{5.66}$$

We denote by $\rho = \beta + i\gamma$ a non-trivial zero of $L(s,\bar{\chi})$ and we choose $T > 0$ to tend to infinity through values such that $|T-\gamma| > \exp(-A_1|\gamma|/\log|\gamma| + 3)$ for every ordinate γ of a zero of $L(s,\chi)$. Using

$$\log|L(s,\chi)| \geqslant \sum_{|t-\gamma|\leqslant 1}\log|t-\gamma| + O(\log(qt))$$

yields

$$\log|L(\sigma+iT,\chi)| \geqslant -\sum_{|T-\gamma|\leqslant 1}A_1\gamma/\log\gamma + O(\log qT) > -A_{\chi}T, \tag{5.67}$$

where $A_{\chi} < \omega$ if A_1 is small enough, and $T > T_0$. Since the main technique behind the proofs of explicit formulae is contour integration, this will enable us to make unwanted horizontal integrals tend to zero as $T \to \infty$ through the above values. To prove that indeed these horizontal integrals tend to zero as $T \to \infty$ for the chosen values we note that from (5.67) we obtain

$$\frac{1}{|L(1-s,\chi)|} = O\left(\exp(A_{\chi}T)\right)$$

where $A_{\chi} < \omega$. Then by Lemma 5.2.2 and Stirling's formula (A.13), we obtain

$$\frac{1}{2\pi i}\int_{\lambda-iT}^{c-iT}\left(\frac{a}{q^{1/2}}\right)^{-s}\frac{Z_1(s)}{L(1-s,\chi)}ds = O\left(\exp\left((A_{\chi}-\omega+\epsilon)|t|\right)\right) \to 0$$

for each $\epsilon > 0$. The other horizontal integral is dealt with similarly.

Consider now $H_F = 0$, then $\nu = 1/2$ and χ is an odd primitive Dirichlet character $\bmod\ q_F$. Therefore $\varphi, \psi \in K(\omega, -\delta)$ is a pair of sine reciprocal functions. Note Z_1 and Z_2 are both analytic at $s = 1$, hence Φ and Ψ both analytic at $s = 1$. Then by the functional equation in Lemma 5.2.2 we see that Φ and Ψ are both analytic at $s = 0$. Therefore both Z_1 and Z_2 are analytic at $s = 0$. Similarly to (5.63), we find that

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}x^{-s}Z_k(s)ds = \begin{cases}\varphi(x) & \text{if } k=1,\\ \psi(x) & \text{if } k=2.\end{cases} \tag{5.68}$$

If χ is an odd, primitive and non-principal character of $\bmod\ q$ then $L(s,\chi)$ satisfies the functional equation

$$\frac{1}{L(1-s,\chi)} = \frac{\tau(\bar{\chi})}{iq^{1/2}}\left(\frac{q}{\pi}\right)^{1/2-s}\frac{\Gamma(1-\frac{s}{2})}{\Gamma(\frac{s+1}{2})}\frac{1}{L(s,\bar{\chi})},$$

for all complex values s. If we use the fact that $ab = 2\pi$ and couple this equation with (5.34), (5.35) and the functional equation of Φ and Ψ in Lemma 5.2.2, then we obtain

$$\frac{1}{2\pi i}\oint_{\Omega}\left(\frac{a}{q^{1/2}}\right)^{-s}\frac{Z_1(s)}{L(1-s,\chi)}ds = \frac{1}{2\pi i}\oint_{\Omega}\frac{\tau(\bar{\chi})}{i(2\pi)^{1/2}}\left(\frac{b}{q^{1/2}}\right)^{s}\frac{Z_2(1-s)}{L(s,\bar{\chi})}ds.$$

By absolute convergence with $\mathrm{Re}(s) = c$ we can change summation and integration to obtain

$$\begin{aligned}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{a}{q^{1/2}}\right)^{-s}\frac{Z_1(s)}{L(1-s,\chi)}ds &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{a}{q^{1/2}}\right)^{-s}\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n^{1-s}}Z_1(s)ds \\ &= \sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n}\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\left(\frac{a}{q^{1/2}n}\right)^{-s}Z_1(s)ds \\ &= \sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n}\varphi\left(\frac{a}{q^{1/2}n}\right),\end{aligned} \tag{5.69}$$

where in ultimate step we have used (5.68) with $k = 1$. Moreover, also by absolute convergence with $\mathrm{Re}(s) = \lambda$, we have

$$\begin{aligned}&\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty}\frac{\tau(\bar{\chi})}{i(2\pi)^{1/2}}\left(\frac{b}{q^{1/2}}\right)^{s}\frac{Z_2(1-s)}{L(s,\bar{\chi})}ds \\ &= \frac{\tau(\bar{\chi})}{i(2\pi)^{1/2}}\frac{1}{2\pi i}\int_{\lambda-i\infty}^{\lambda+i\infty}\left(\frac{b}{q^{1/2}}\right)^{s}\sum_{n=1}^{\infty}\frac{\bar{\chi}(n)\mu(n)}{n^s}Z_2(1-s)ds \\ &= \frac{\tau(\bar{\chi})}{i(2\pi)^{1/2}}\frac{b}{q^{1/2}}\sum_{n=1}^{\infty}\frac{\bar{\chi}(n)\mu(n)}{n}\frac{1}{2\pi i}\int_{1-\lambda-i\infty}^{1-\lambda+i\infty}\left(\frac{b}{q^{1/2}n}\right)^{-w}Z_2(w)dw \\ &= \frac{\tau(\bar{\chi})}{i(2\pi)^{1/2}}\frac{b}{q^{1/2}}\sum_{n=1}^{\infty}\frac{\bar{\chi}(n)\mu(n)}{n}\psi\left(\frac{b}{q^{1/2}n}\right),\end{aligned}$$

where we have made the change $w = 1 - s$. A similar reasoning as we used for even primitive characters shows that the contribution from the horizontal integrals of this contour will tend to zero as well. Next, we compute the residues

- for the non-trivial zeros ρ one has

$$\sum_{\rho}\operatorname*{Res}_{s=\rho}\frac{\tau(\bar{\chi})}{i(2\pi)^{1/2}}\left(\frac{b}{q^{1/2}}\right)^{s}\frac{Z_2(1-s)}{L(s,\bar{\chi})} = \frac{\tau(\bar{\chi})}{i(2\pi)^{1/2}}\sum_{\rho}\left(\frac{b}{q^{1/2}}\right)^{\rho}\frac{Z_2(1-\rho)}{L'(\rho,\bar{\chi})}.$$

By the residue theorem one has

$$\begin{aligned}&\frac{\tau(\bar{\chi})}{i(2\pi)^{1/2}}\frac{b}{q^{1/2}}\sum_{n=1}^{\infty}\frac{\bar{\chi}(n)\mu(n)}{n}\psi\left(\frac{b}{q^{1/2}n}\right) - \sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n}\varphi\left(\frac{a}{q^{1/2}n}\right) \\ &= \frac{\tau(\bar{\chi})}{i(2\pi)^{1/2}}\sum_{\rho}\left(\frac{b}{q^{1/2}}\right)^{\rho}\frac{Z_2(1-\rho)}{L'(\rho,\bar{\chi})}.\end{aligned}$$

Multiplying by $-\sqrt{a}\sqrt{\tau(\chi)}$ and using the fact that $\sqrt{\tau(\chi)\tau(\bar{\chi})} = iq^{1/2}$ one has

$$\begin{aligned}&\sqrt{a}\sqrt{\tau(\chi)}\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n}\varphi\left(\frac{a}{q^{1/2}n}\right) - \sqrt{b}\sqrt{\tau(\bar{\chi})}\sum_{n=1}^{\infty}\frac{\bar{\chi}(n)\mu(n)}{n}\psi\left(\frac{b}{q^{1/2}n}\right) \\ &= -\frac{q^{1/2}}{b^{1/2}}\sqrt{\tau(\bar{\chi})}\sum_{\rho}\left(\frac{b}{q^{1/2}}\right)^{\rho}\frac{\Gamma(1-\rho)}{L'(\rho,\bar{\chi})}Z_2(1-\rho),\end{aligned} \tag{5.70}$$

and this proves the theorem. □

Proof of Corollary (5.19). By taking $\nu = \frac{1}{2}$ so that $\chi(-1) = -2\frac{1}{2} = -1$, and choosing

$$\varphi(x) = \frac{1}{e^{\sqrt{2\pi x}} - 1} - \frac{1}{\sqrt{2\pi x}},$$

we have

$$\begin{aligned}\psi(x) &= \int_0^\infty (ux)^{\frac{1}{2}} J_\nu(ux)\varphi(u)du = \sqrt{\frac{2}{\pi}}\int_0^\infty \sin(ux)\varphi(u)du \\ &= -\frac{1}{2} - \frac{1}{\sqrt{2\pi x}} + \frac{1}{2}\coth\left(\sqrt{\frac{\pi}{2}}x\right) = \frac{1}{e^{\sqrt{2\pi x}} - 1} - \frac{1}{\sqrt{2\pi x}} = \varphi(x).\end{aligned}$$

The Mellin transform is given (see §9.12 of [Tit48] and equation (2.7.1) of [Tit86])

$$Z_i(s) = \int_0^\infty x^{s-1}\left(\frac{1}{e^{\sqrt{2\pi x}} - 1} - \frac{1}{\sqrt{2\pi x}}\right)dx = (2\pi)^{-\frac{1}{2}s}\Gamma(s)\zeta(s),$$

for $0 < \mathrm{Re}(s) < 1$ and $i = 1, 2$. We note that

$$Z_i(1-\rho) = (2\pi)^{-\frac{1}{2}(1-\rho)}\Gamma(1-\rho)\zeta(1-\rho) = 0.$$

By inserting these into (5.18), we obtain

$$\begin{aligned}&\sqrt{a\tau(\chi)}\sum_{n=1}^\infty \frac{\chi(n)\mu(n)}{n}\left(\frac{1}{e^{a\sqrt{2\pi/qn}} - 1} - \frac{n}{a}\sqrt{\frac{q}{2\pi}}\right) \\ &- \sqrt{b\tau(\bar{\chi})}\sum_{n=1}^\infty \frac{\bar{\chi}(n)\mu(n)}{n}\left(\frac{1}{e^{b\sqrt{2\pi/qn}} - 1} - \frac{n}{b}\sqrt{\frac{q}{2\pi}}\right) \\ &= \sqrt{\frac{q\tau(\chi)}{2\pi a}}\sum_{\rho\in B_\chi}\left(\frac{(2\pi)^{1/2}a}{q^{1/2}}\right)^\rho \frac{\Gamma(1-\rho)\zeta(1-\rho)}{L'(\rho,\chi)},\end{aligned}$$

as was to be shown. □

Proof of Corollary 5.1.3. First take χ to be even, i.e. $1 = \chi(-1) = -2\nu$ so that $\nu = -\frac{1}{2}$. Choose $\varphi(x) = \mathrm{sech}(\frac{1}{\sqrt{2}}\sqrt{\pi}x)$. We verify that this is cosine reciprocal by noting that

$$\begin{aligned}\psi(x) &= \int_0^\infty (ux)^{\frac{1}{2}} J_{-\frac{1}{2}}(ux)\varphi(u)du = \sqrt{\frac{2}{\pi}}\int_0^\infty \cos(ux)\varphi(u)du \\ &= \sqrt{\frac{2}{\pi}}\int_0^\infty \cos(ux)\,\mathrm{sech}(\tfrac{1}{\sqrt{2}}\sqrt{\pi}u)du = \mathrm{sech}(\tfrac{1}{\sqrt{2}}\sqrt{\pi}x) = \varphi(x).\end{aligned}$$

The Mellin transform is given (see entry 6.1 of [Obe74]) by

$$Z_i(s) = 2^{1-\frac{3}{2}s}\pi^{-\frac{s}{2}}\Gamma(s)(\zeta(s,\tfrac{1}{4}) - \zeta(s,\tfrac{3}{4}))$$

for $\mathrm{Re}(s) > 0$ and $i = 1, 2$. Inserting this into (5.18) we obtain

$$\sqrt{a\tau(\chi)}\sum_{n=1}^\infty \frac{\chi(n)\mu(n)}{n}\,\mathrm{sech}\left(\sqrt{\frac{\pi}{2q}}\frac{a}{n}\right) - \sqrt{b\tau(\bar{\chi})}\sum_{n=1}^\infty \frac{\bar{\chi}(n)\mu(n)}{n}\left(\sqrt{\frac{\pi}{2q}}\frac{b}{n}\right)$$

$$= \sqrt{\frac{q\tau(\chi)}{2\pi a}} \sum_{\rho\in B_\chi} \left(\frac{2^{\frac{3}{2}}\pi^{\frac{1}{2}}a}{q^{1/2}}\right)^{\rho} \frac{\Gamma(1-\rho)(\zeta(1-\rho,\frac{1}{4})-\zeta(1-\rho,\frac{3}{4}))}{L'(\rho,\chi)}.$$

Next, take the same choice of φ and insert it into (5.17) so that

$$\sqrt{a}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\operatorname{sech}\left(\sqrt{\frac{\pi}{2}}\frac{a}{n}\right) - \sqrt{b}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\operatorname{sech}\left(\sqrt{\frac{\pi}{2}}\frac{a}{n}\right)$$
$$= \sqrt{\frac{1}{2\pi a}} \sum_{\rho\in B} (2^{\frac{3}{2}}\pi^{\frac{1}{2}}a)^{\rho} \frac{\Gamma(1-\rho)(\zeta(1-\rho,\frac{1}{4})-\zeta(1-\rho,\frac{3}{4}))}{\zeta'(\rho)}$$

and this ends the proof. □

Proof of Corollary 5.1.4. It is known from §1.1 of [HT31] that

$$f(x) = x^{\frac{1}{2}}J_{-\frac{1}{4}}(\tfrac{1}{2}x^2)$$

is a cosine reciprocal function, i.e. if $\varphi(x) = f(x)$ and if we take $\nu = -\frac{1}{2}$ then $\psi(x) = \varphi(x)$. The Mellin transform is given by

$$Z_i(s) = \int_0^\infty x^{s-1}\varphi(x)dx = 2^{-\frac{3}{2}+s}\frac{\Gamma(\frac{s}{4})}{\Gamma(\frac{3}{4}-\frac{s}{4})},$$

for $i = 1, 2$ and $0 < \mathrm{Re}(s) < \frac{3}{2}$. Inserting this back into (5.17) yields

$$a\sum_{n=1}^{\infty}\frac{\mu(n)}{n^{3/2}}J_{-\frac{1}{4}}\left(\frac{1}{2}\frac{a^2}{n^2}\right) - b\sum_{n=1}^{\infty}\frac{\mu(n)}{n^{3/2}}J_{-\frac{1}{4}}\left(\frac{1}{2}\frac{a^2}{n^2}\right) = \frac{1}{\sqrt{2a}}\sum_{\rho\in B}\left(\frac{a}{2}\right)^{\rho}\frac{1}{\zeta'(\rho)}\frac{\Gamma(\frac{1}{4}-\frac{\rho}{4})}{\Gamma(\frac{1}{2}+\frac{\rho}{4})}.$$

Next, for $\nu = \pm\frac{1}{2}$ set

$$\varphi_\nu(x) = x^{\frac{1}{2}}J_{\frac{\nu}{2}}(\tfrac{1}{2}x^2).$$

If $\nu = -\frac{1}{2}$, then

$$\varphi_{-\frac{1}{2}}(x) = x^{\frac{1}{2}}J_{-\frac{1}{4}}(\tfrac{1}{2}x^2) = \psi(x),$$

as explained in the beginning of the proof. In this case, (5.18) yields

$$\frac{a\sqrt{\tau(\chi)}}{q^{1/4}}\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n^{3/2}}J_{-\frac{1}{4}}\left(\frac{a^2}{2qn^2}\right) - \frac{b\sqrt{\tau(\bar\chi)}}{q^{1/4}}\sum_{n=1}^{\infty}\frac{\bar\chi(n)\mu(n)}{n^{3/2}}J_{-\frac{1}{4}}\left(\frac{b^2}{2qn^2}\right)$$
$$= \sqrt{\frac{q\tau(\chi)}{2a}}\sum_{\rho\in B_\chi}\left(\frac{a}{2q^{1/2}}\right)^{\rho}\frac{1}{L'(\rho,\chi)}\frac{\Gamma(\frac{1}{4}-\frac{\rho}{4})}{\Gamma(\frac{1}{2}+\frac{\rho}{4})}.$$

On the other hand, if $\nu = \frac{1}{2}$, then

$$\psi_{\frac{1}{2}}(x) = \int_0^\infty (ux)^{\frac{1}{2}}J_\nu(ux)\varphi_{\frac{1}{2}}(u)du = \sqrt{\frac{2}{\pi}}\int_0^\infty \sin(ux)u^{\frac{1}{2}}J_{\frac{1}{4}}(\tfrac{1}{2}u^2)du$$
$$= \frac{x}{\sqrt{2}\Gamma(\frac{5}{4})}{}_0F_1(\tfrac{5}{4},-\tfrac{x^4}{16}) = x^{\frac{1}{2}}J_{\frac{1}{4}}(\tfrac{1}{2}x^2) = \varphi_{\frac{1}{2}}(x),$$

where ${}_0F_1$ is the confluent hypergeometric limit function (A.28). Therefore, plugging back into (5.18) gives us

$$\frac{a\sqrt{\tau(\chi)}}{q^{1/4}}\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n^{3/2}}J_{\frac{1}{4}}\left(\frac{a^2}{2qn^2}\right)-\frac{b\sqrt{\tau(\bar{\chi})}}{q^{1/4}}\sum_{n=1}^{\infty}\frac{\bar{\chi}(n)\mu(n)}{n^{3/2}}J_{\frac{1}{4}}\left(\frac{b^2}{2qn^2}\right)$$
$$=\sqrt{\frac{q\tau(\chi)}{2a}}\sum_{\rho\in B_\chi}\left(\frac{a}{2q^{1/2}}\right)^{\rho}\frac{1}{L'(\rho,\chi)}\frac{\Gamma(\frac{1}{4}-\frac{\rho}{4})}{\Gamma(\frac{1}{2}+\frac{\rho}{4})}.$$

Therefore, combining both cases the proof is finished. □

Proof of Corollary 5.1.5. In [Phi36] it is shown that for $\mathrm{Re}(a) > 0$

$$x^{\frac{1}{2}+\mu}(x^2+a^2)^{\frac{1}{4}(-\mu-1)}K_{\frac{1}{2}(\mu+1)}(a\sqrt{x^2+a^2})$$

is Hankel reciprocal with respect to μ, where $K_\nu(z)$ is the K-Bessel function (A.18). Its Mellin transform is given by

$$\phi_\mu(s)=\int_0^\infty x^{s+\mu-\frac{1}{2}}(x^2+a^2)^{-\frac{1}{4}(\mu+1)}K_{\frac{1}{2}(\mu+1)}(a\sqrt{x^2+a^2})dx$$
$$=2^{\frac{1}{2}s+\frac{1}{2}\mu-\frac{3}{4}}\Gamma(\tfrac{1}{2}s+\tfrac{1}{2}\mu+\tfrac{1}{4})K_{-\frac{1}{2}(s-\frac{1}{2})}(a^2).$$

If we take $\mu=-\frac{1}{2}$ and use cosine reciprocity, then

$$(x^2+a^2)^{-\frac{1}{8}}K_{\frac{1}{4}}(a\sqrt{a^2+x^2})$$

is cosine reciprocal. Thus,

$$Z_1(1-\rho)=\phi_{-\frac{1}{2}}(1-\rho)=2^{\frac{1}{2}(1-\rho)-1}\Gamma(\tfrac{1}{2}(1-\rho))K_{-\frac{1}{2}(\frac{1}{2}-\rho)}(z^2)$$

Inserting these back into (5.17) gives us

$$\sqrt{a}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\left(\frac{a^2}{n^2}+z^2\right)^{-\frac{1}{8}}K_{\frac{1}{4}}\left(z\sqrt{z^2+\frac{a^2}{n^2}}\right)$$
$$-\sqrt{b}\sum_{n=1}^{\infty}\frac{\mu(n)}{n}\left(\frac{b^2}{n^2}+z^2\right)^{-\frac{1}{8}}K_{\frac{1}{4}}\left(z\sqrt{z^2+\frac{b^2}{n^2}}\right)$$
$$=\frac{1}{\sqrt{2a}}\sum_{\rho\in B}\left(\frac{a}{2^{1/2}}\right)^{\rho}\frac{\Gamma(\frac{1-\rho}{2})K_{-\frac{1}{2}(\frac{1}{2}-\rho)}(z^2)}{\zeta'(\rho)}$$

and (5.18) gives us

$$\sqrt{a\tau(\chi)}\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n}\left(\frac{a^2}{qn^2}+z^2\right)^{-\frac{1}{8}}K_{\frac{1}{4}}\left(z\sqrt{z^2+\frac{a^2}{qn^2}}\right)$$

$$- \sqrt{b\tau(\bar{\chi})} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)\mu(n)}{n} \left(\frac{b^2}{qn^2} + z^2 \right)^{-\frac{1}{8}} K_{\frac{1}{4}} \left(z \sqrt{z^2 + \frac{b^2}{qn^2}} \right)$$
$$= \sqrt{\frac{q\tau(\chi)}{2a}} \sum_{\rho \in B_\chi} \left(\frac{a}{q^{1/2}2^{1/2}} \right)^{\rho} \frac{\Gamma(\frac{1-\rho}{2}) K_{-\frac{1}{2}(\frac{1}{2}-\rho)}(z^2)}{L'(\rho, \chi)}.$$

If we take $\mu = \frac{1}{2}$ then the same procedure on ϕ gives

$$Z_1(1-\rho) = \phi_{\frac{1}{2}}(1-\rho) = 2^{-\frac{1}{2}\rho} \Gamma(1 - \tfrac{1}{2}\rho) K_{-\frac{1}{2}(\frac{1}{2}-\rho)}(z^2).$$

Therefore (5.18) yields

$$\frac{a}{q^{1/2}} \sqrt{a\tau(\chi)} \sum_{n=1}^{\infty} \frac{\chi(n)\mu(n)}{n^2} \left(\frac{a^2}{qn^2} + z^2 \right)^{-\frac{3}{8}} K_{\frac{3}{4}} \left(z \sqrt{z^2 + \frac{a^2}{qn^2}} \right)$$
$$- \frac{b}{q^{1/2}} \sqrt{b\tau(\bar{\chi})} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)\mu(n)}{n^2} \left(\frac{b^2}{qn^2} + z^2 \right)^{-\frac{3}{8}} K_{\frac{3}{4}} \left(z \sqrt{z^2 + \frac{b^2}{qn^2}} \right)$$
$$= \sqrt{\frac{q\tau(\chi)}{a}} \sum_{\rho \in B_\chi} \left(\frac{a}{2^{1/2}q^{1/2}} \right)^{\rho} \frac{\Gamma(1 - \frac{1}{2}\rho) K_{-\frac{1}{2}(\frac{1}{2}-\rho)}(z^2)}{L'(\rho, \chi)}.$$

Combining both cases yields the corollary. □

Proof of Corollary 5.1.6. In [Var37] it is shown that

$$x^{\nu+1/2} e^{x^2/4} D_{-2\nu-3}(x) = \int_0^{\infty} (xy)^{\frac{1}{2}} J_\nu(xy) y^{\nu+1/2} e^{y^2/4} D_{-2\nu-3}(y) dy$$

for $\mathrm{Re}(\nu) > -1$, and that if

$$f(s) = \int_0^{\infty} x^{s-1} e^{x^2/4} D_n(x) dx = \frac{\Gamma(s)\Gamma(-\frac{1}{2}n - \frac{1}{2}s)}{2^{n/2+s/2+1}\Gamma(-n)},$$

then

$$f(s+\lambda) = 2^{s-1/2} \frac{\Gamma(\frac{1}{2}\nu + \frac{1}{2}s + \frac{1}{4})}{\Gamma(\frac{1}{2}\nu - \frac{1}{2}s + \frac{3}{4})} f(\lambda + 1 - s).$$

Next, take $\nu = -\frac{1}{2}$ so that we have

$$\varphi(x) = e^{x^2/4} D_{-2}(x) = \psi(x),$$

and

$$Z_1(s) = \int_0^{\infty} \varphi(x) x^{s-1} dx = \int_0^{\infty} x^{s-1} e^{x^2/4} D_{-2}(x) dx = \frac{\Gamma(s)\Gamma(1 - \frac{1}{2}s)}{2^{s/2}}.$$

Substitute this in (5.17) to obtain

$$\sqrt{a} \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \exp\left(\frac{a^2}{4n^2} \right) D_{-2}\left(\frac{a}{n} \right) - \sqrt{b} \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \exp\left(\frac{b^2}{4n^2} \right) D_{-2}\left(\frac{b}{n} \right)$$

$$= \frac{1}{2^{1/2}a^{1/2}} \sum_{\rho \in B} (2^{1/2}a)^{\rho} \frac{\Gamma(1-\rho)\Gamma(\frac{1}{2}+\frac{1}{2}\rho)}{\zeta'(\rho)}.$$

Substituting the above in (5.18) gives us

$$\begin{aligned}
&\sqrt{a\tau(\chi)} \sum_{n=1}^{\infty} \frac{\chi(n)\mu(n)}{n} \exp\left(\frac{a^2}{4qn^2}\right) D_{-2}\left(\frac{a}{q^{1/2}n}\right) \\
&- \sqrt{b\tau(\bar{\chi})} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)\mu(n)}{n} \exp\left(\frac{b^2}{4qn^2}\right) D_{-2}\left(\frac{b}{q^{1/2}n}\right) \\
&= \sqrt{\frac{q\tau(\chi)}{2a}} \sum_{\rho \in B_{\chi}} \left(\frac{2^{1/2}a}{q^{1/2}}\right)^{\rho} \frac{\Gamma(1-\rho)\Gamma(\frac{1}{2}+\frac{1}{2}\rho)}{L'(\rho,\chi)}
\end{aligned}$$

Finally, taking instead $\nu = \frac{1}{2}$ yields

$$\varphi(x) = xe^{x^2/4} D_{-4}(x) = \psi(x)$$

and

$$\begin{aligned}
Z_1(s) &= \int_0^{\infty} \varphi(x) x^{s-1} dx = \int_0^{\infty} x^{s-1+1} e^{x^2/4} D_{-4}(x) dx \\
&= f(s+1) = 2^{s-1/2} \frac{\Gamma(\frac{1}{2}+\frac{1}{2}s)}{\Gamma(1-\frac{1}{2}s)} f(2-s) \\
&= 2^{3s/2-1/2} \frac{\Gamma(\frac{1}{2}+\frac{1}{2}s)}{\Gamma(1-\frac{1}{2}s)} \frac{\Gamma(2-s)\Gamma(1+\frac{1}{2}s)}{6} = \frac{2^{(s-1)/2}\pi^{3/2}(s-1)\csc(\pi s)}{3\Gamma(-\frac{s}{2})} \qquad (5.71)
\end{aligned}$$

Thus, substituting this in (5.18) yields

$$\begin{aligned}
&a\sqrt{\frac{a\tau(\chi)}{q}} \sum_{n=1}^{\infty} \frac{\chi(n)\mu(n)}{n^2} \exp\left(\frac{a^2}{4qn^2}\right) D_{-4}\left(\frac{a}{q^{1/2}n}\right) \\
&- b\sqrt{\frac{b\tau(\bar{\chi})}{q}} \sum_{n=1}^{\infty} \frac{\bar{\chi}(n)\mu(n)}{n^2} \exp\left(\frac{b^2}{4qn^2}\right) D_{-4}\left(\frac{b}{q^{1/2}n}\right) \\
&= \frac{1}{3}\sqrt{\frac{q\tau(\chi)}{a}} \sum_{\rho \in B_{\chi}} \left(\frac{a}{2^{3/2}q^{1/2}}\right)^{\rho} \frac{\Gamma(1-\frac{1}{2}\rho)\Gamma(1+\rho)\Gamma(\frac{3}{2}-\frac{1}{2}\rho)}{\Gamma(\frac{1}{2}+\frac{1}{2}\rho)L'(\rho,\chi)}
\end{aligned}$$

and this ends the proof. □

Proof of Corollary 5.1.7. From [Var37] we know that

$$x^{\nu-\frac{1}{2}} e^{-\frac{1}{4}x^2} D_{-2\nu}(x) = \int_0^{\infty} (xy)^{\frac{1}{2}} J_{\nu}(xy) y^{\nu-\frac{1}{2}} e^{-\frac{1}{4}y^2} D_{-2\nu}(y) dy.$$

Thus, if we take $\nu = \frac{1}{2}$ and note that

$$\int_0^{\infty} x^{s-1} e^{\frac{1}{4}x^2} D_{-1}(x) dx = \frac{\Gamma(s)\Gamma(\frac{1}{2}-\frac{1}{2}s)}{2^{s/2}2^{1/2}},$$

then (5.18) yields

$$\begin{aligned}
&\sqrt{a\tau(\chi)}\sum_{n=1}^{\infty}\frac{\chi(n)\mu(n)}{n}\exp\left(\frac{a^2}{4qn^2}\right)D_{-1}\left(\frac{a}{q^{1/2}n}\right)\\
&-\sqrt{b\tau(\bar{\chi})}\sum_{n=1}^{\infty}\frac{\bar{\chi}(n)\mu(n)}{n}\exp\left(\frac{b^2}{4qn^2}\right)D_{-1}\left(\frac{b}{q^{1/2}n}\right)\\
&=\frac{1}{2}\sqrt{\frac{q\tau(\chi)}{a}}\sum_{\rho\in B_\chi}\left(\frac{2^{1/2}a}{q^{1/2}}\right)^{\rho}\frac{\Gamma(1-\rho)\Gamma(\frac{1}{2}\rho)}{L'(\rho,\chi)},
\end{aligned}$$

as desired. □

Chapter 6

On a mollifier of the perturbed Riemann zeta-function

6.1 Introduction

This chapter is based on a preprint [KRZ16], in collaboration with N. Robles and D. Zeindler.

6.1.1 Statement of the results

The functional equation of the Riemann zeta-function can be expressed in terms of the ξ-function as

$$\xi(s) = \xi(1-s),$$

where

$$\xi(s) = H(s)\zeta(s) \quad \text{and} \quad H(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma\left(\frac{s}{2}\right).$$

If $N(T)$ denotes the number of the complex zeros of $\xi(s)$ up to height $0 \le \gamma < T$ then

$$N(T) = \frac{T}{2\pi}\left(\log\frac{T}{2\pi} - 1\right) + \frac{7}{8} + S(T) + O\left(\frac{1}{T}\right),$$

where

$$S(T) = \frac{1}{\pi}\arg\zeta\left(\frac{1}{2}+it\right) \ll \log T,$$

as $T \to \infty$, see e.g. [MV07; Tit86] for properties of $\zeta(s)$. To state the results, we let $N_0(T)$ denote the number of non-trivial zeros up to height $T > 0$ such that $\beta = 1/2$. Similarly, let $N_0^*(T)$ denote the number such zeros which are also simple. We then define

$$\kappa = \liminf_{T\to\infty}\frac{N_0(T)}{N(T)} \quad \text{and} \quad \kappa^* = \liminf_{T\to\infty}\frac{N_0^*(T)}{N(T)}.$$

The history behind the value of κ can be found in [BCY11; Fen12; RRZ16]. The main breakthroughs were as follows. In 1942, Selberg [Sel42] established that $0 < \kappa \le 1$. Levinson later showed in 1974 that $\kappa \ge .3474$. This was improved by Conrey to $\kappa \ge .4088$ in 1989 and later refined by Bui, Conrey and Young [BCY11] to $\kappa \ge .4105$, and shortly afterward by Feng [Fen12] to $\kappa \ge .4127$. It should be noted that both results are improvements of $\kappa \ge .4088$ and are independent of each other.

Robles, Roy and Zaharescu [RRZ16] as well as Bui [Bui14] brought up a point regarding the strength of Feng's result. In [RRZ16], it was explained that $\kappa \ge .4107$, unconditionally, using Feng's mollifier. However, the computation of the mixed terms of the mollifiers of Conrey and of Feng was not carried through explicitly.

In this chapter, we close this gap and we explain Feng's brilliant choice in the context of the powerful technology developed in [BCY11; You10]. These ideas come from the ratios conjectures of L-functions due to Conrey, Farmer and Zirnbaeuer [JZ08] as well as to Conrey and Snaith [CS07]. It should be noted that Feng's methodology to obtain the main terms of his theorem consisted of an ingenious combination of elementary methods, namely induction and Mertens' formula, applied to Conrey's result [Con89]. On the other hand, this choice of methods blurred a bit the length the mollifier was allowed to take.

Lastly, the closing of this gap will clarify the situation of the percentage of non-trivial on the critical line when one attaches Feng's second-piece mollifier to Conrey's.

6.1.2 Choice of mollifiers

Let $Q(x)$ be a real polynomial satisfying $Q(0) = 1$, $Q(x) + Q(1 - x) = \text{constant}$, and define

$$V(s) = Q\Big(-\frac{1}{L}\frac{d}{ds}\Big)\zeta(s), \tag{6.1}$$

where for large T,

$$L = \log T.$$

We recall that a mollifier is a finite Dirichlet series

$$\psi(s) = \sum_{n\leq y} \frac{b(n)}{n^s}$$

that approximates a certain meromorphic function. If $\psi(s)$ is a mollifier, then it is well-known from the work of Levinson [Lev74] and of Conrey [Con89] that Littlewood's lemma (see [Tit86, §9.9]) followed by the arithmetic and geometric mean inequalities yields

$$\kappa \geq 1 - \frac{1}{R}\log\left(\frac{1}{T}\int_1^T |V\psi(\sigma_0 + it)|^2 dt\right) + o(1), \tag{6.2}$$

where $\sigma_0 = 1/2 - R/L$, and R is a bounded positive real number to be chosen later. Following Feng [Fen12], we will choose a mollifier of the form

$$\psi(s) = \psi_1(s) + \psi_2(s),$$

where ψ_1 is the mollifier considered by Conrey in [Con89]. Let $P_1(x) = \sum_j a_j x^j$ be a certain polynomial satisfying $P_1(0) = 0$, $P_1(1) = 1$, and let $y_1 = T^{\theta_1}$ where $0 < \theta_1 < 1/2$. We adopt the notation

$$P_1[n] = P_1\left(\frac{\log(y_1/n)}{\log y_1}\right)$$

for $1 \le n \le y_1$. By convention, we set $P_1[x] = 0$ for $x \ge y_1$. Then $\psi_1(s)$ is given by

$$\psi_1(s) = \sum_{h \le y_1} \frac{\mu(h) h^{\sigma_0 - 1/2}}{h^s} P_1[h], \tag{6.3}$$

where $\mu(n)$ is the Möbius function. For the second mollifier, we take

$$\psi_2(s) = \sum_{k \leqslant y_2} \frac{\mu(k) k^{\sigma_0 - 1/2}}{k^s} \sum_{\ell=2}^{K} \sum_{p_1 \cdots p_\ell | k} \frac{\log p_1 \cdots \log p_\ell}{\log^\ell y_2} P_\ell[k]. \tag{6.4}$$

Here $K \ge 2$ is an integer of our choice and $p_1, \ldots, p_\ell$ are distinct primes. Also we need $P_\ell(0) = 0$ for $\ell = 2, \cdots, K$. In this case $y_2 = T^{\theta_2}$ where $0 < \theta_2 < 1/2$.

Remark 6.1.1. *It will become clear in the calculation of the crossterm integral between ψ_1 and ψ_2 that one needs $\theta_1 + \theta_2 < 1 - \varepsilon$. Therefore, if θ_1 increases, then θ_2 decreases unless some difficult work is done to push θ_2 back to its original (or higher) value. See the comments between Theorem 6.1.1 and Theorem 6.1.2 for more details.*

The reason behind this choice of $\psi_2(s)$ is that we wish to mollify not only $\zeta(s)$ but also $\frac{\zeta'(s)}{\log T}$, which is the second term coming from (6.1). This is accomplished by looking at

$$\frac{1}{\zeta(s) + \frac{\zeta'(s)}{\log T}} = \frac{1}{\zeta(s)} - \frac{1}{\log T} \frac{\zeta'}{\zeta^2}(s) + \frac{1}{\log^2 T} \frac{(\zeta')^2}{\zeta^3}(s) - \frac{1}{\log^3 T} \frac{(\zeta')^3}{\zeta^4}(s) + \cdots. \tag{6.5}$$

When k is a square-free positive integer, then one has

$$(\mu * \Lambda^{*\ell})(k) = (-1)^\ell \mu(k) \sum_{p_1 \cdots p_\ell | k} \log p_1 \cdots \log p_\ell,$$

where $f * g$ denotes the Dirichlet convolution of arithmetic functions f and g. One can then notice that the mollifier (6.4) is approximating (6.5) up to K terms. Here $\Lambda^{*\ell}$ stands for convolving the von Mangoldt function $\Lambda(n)$ with itself exactly ℓ times. If k contains a square divisor, then, as remarked by Feng [Fen12], the coefficients a_j resulting from (6.5) contribute a lower order to the mean value integrals I_{11}, I_{12} and I_{22} related to κ in (6.2) (see below for exact definitions of these I-integrals).

6.1.3 Numerical evaluations

We will prove the following numerical results.

Theorem 6.1.1. *We have*

$$\kappa \ge .369927 \quad \text{and} \quad \kappa^* \ge .359991,$$

unconditionally.

Using the work of Iwaniec and Deshouillers [DI82; DI84], Conrey [Con89] was able to push the size of the mollifier ψ_1 to $\theta_1 < 4/7 - \varepsilon$. In the light of Lemma 6.2.1 and (6.23) below, we require $\theta_1 + \theta_2 < 1$ in our argumentation. The points brought up in [Bui14] and [RRZ16] show that some difficult work is needed if one takes $\theta_1 + \theta_2 > 1$. Theorem 6.1.1 utilizes $\theta_1, \theta_2 < 1/2 - \varepsilon$. However, if we take $\theta_1 < 4/7 - \varepsilon$ and $\theta_2 < 3/7 - \varepsilon$, then we get

Theorem 6.1.2. *We have*

$$\kappa \geq .410725 \quad \text{and} \quad \kappa^* \geq .403211,$$

unconditionally.

It should therefore be stressed that Theorem 6.1.1 is an improvement of the last theorem to ever use $\theta_1 = 1/2 - \varepsilon$, namely the first corollary of [Con83], where it was shown that $\kappa \geq .3658$.

The method sketched in [BCY11; RRZ16] deals with multiple piece mollifiers and our main result used to prove Theorem 6.1.1 and Theorem 6.1.2 reads as follows.

Theorem 6.1.3. *Suppose that $\theta_1 + \theta_2 = 1 - \varepsilon$ with $\theta_1 < 4/7$ and $\theta_2 < 1/2$ and $\varepsilon > 0$ small. Then*

$$\frac{1}{T}\int_1^T |V\psi(\sigma_0 + it)|^2 dt = c(P_1, P_\ell, Q, R, \theta_1, \theta_2) + o(1),$$

where $c(P_1, P_\ell, Q, R, \theta_1, \theta_2) = c_{11} + 2c_{12} + c_{22}$ and the c_{ij} are given by (6.6), (6.7) and (6.8).

We use `Mathematica` to numerically evaluate $c(P_1, P_\ell, Q, R, 1/2, 1/2)$ with the following choices of parameters. With $K = 3$, $R = 1.3$,

$$\begin{aligned}
Q(x) &= .481936 + .632349(1-2x) - .144698(1-2x)^3 + .0304136(1-2x)^5,\\
P_1(x) &= x + .225339x(1-x) - 1.01137x(1-x)^2 + .174004x(1-x)^3\\
&\quad - .100235x(1-x)^4,\\
P_2(x) &= 1.05138x + .284201x^2,\\
P_3(x) &= .222032x - .13254x^2,
\end{aligned}$$

we have $\kappa \geq .369927$. To obtain $\kappa^* \geq .359991$, we take $K = 3$, $R = 1.2$,

$$\begin{aligned}
Q(x) &= 0.476202 + .523798(1-2x),\\
P_1(x) &= x + .0531913x(1-x) - .594999x(1-x)^2 - .00107597x(1-x)^3\\
&\quad - .0761954x(1-x)^4,\\
P_2(x) &= .896567x - .0297464x^2,\\
P_3(x) &= .0699271x - .108964x^2.
\end{aligned}$$

We also use `Mathematica` to numerically evaluate $c(P_1, P_\ell, Q, R, 4/7, 3/7)$ with the following choices of parameters. With $K = 3$, $R = 1.295$,

$$\begin{aligned}
Q(x) &= .492203 + .621972(1-2x) - .148163(1-2x)^3 + .033988(1-2x)^5,\\
P_1(x) &= x + .229117x(1-x) - 2.932318x(1-x)^2 + 4.856163x(1-x)^3\\
&\quad - 2.390999x(1-x)^4\\
P_2(x) &= -.072644x + 1.559440x^2\\
P_3(x) &= .701568x - .554403x^2
\end{aligned}$$

we have $\kappa \geq .410725$. To obtain $\kappa^* \geq .403211$, we take $K = 3$, $R = 1.109$,

$$\begin{aligned}
Q(x) &= .485034 + .514966(1-2x),\\
P_1(x) &= x + .0486916x(1-x) - 2.02526x(1-x)^2 + 3.43611x(1-x)^3
\end{aligned}$$

$$
\begin{aligned}
&\quad - 1.62355x(1-x)^4,\\
P_2(x) &= -.034431x + 1.09223x^2,\\
P_3(x) &= .479296x - 0.385868x^2.
\end{aligned}
$$

It is interesting to see how the second piece ψ_2 of Feng contributes to the % at its "natural" size $\theta_2 < 1/2 - \varepsilon$ and this was not remarked before in the literature. While it adds .19% at $\theta_2 < 3/7 - \varepsilon$, it adds .4127% at $\theta_2 < 1/2 - \varepsilon$. Naturally, since ψ_2 is the perturbation of ψ_1, it behooves us to take θ_1 as large as possible, in this case $\theta_1 < 4/7 - \varepsilon$.

6.1.4 The smoothing argument

The idea of smoothing the mean value integrals was introduced in [BCY11; You10] and it helps substantially in our calculations. Let $w(t)$ be a smooth function satisfying the following properties:

(a) $0 \le w(t) \le 1$ for all $t \in \mathbb{R}$,

(b) w has compact support in $[T/4, 2T]$,

(c) $w^{(j)}(t) \ll_j \Delta^{-j}$, for each $j = 0, 1, 2, \dots$ and where $\Delta = T/L$.

This allows us to re-write Theorem 6.1.3 as follows.

Theorem 6.1.4. *Suppose that $\theta_1 + \theta_2 = 1 - \varepsilon$ with $\theta_1 < 4/7$ and $\theta_2 < 1/2$ and $\varepsilon > 0$ small. For any w satisfying conditions* (a), (b) *and* (c) *and $\sigma_0 = 1/2 - R/L$,*

$$\int_{-\infty}^{\infty} w(t)|V\psi(\sigma_0 + it)|^2 dt = c(P_1, P_\ell, Q, R, \theta_1, \theta_2)\widehat{w}(0) + O(T/L),$$

uniformly for $R \ll 1$, where $c(P_1, P_\ell, Q, R, \theta_1, \theta_2) = c_{11} + 2c_{12} + c_{22}$ and the c_{ij} are given by (6.6), (6.7) *and* (6.8).

How to deal with a two-piece mollifier was explained in [BCY11; Fen12]. In [RRZ16] a 4-piece mollifier was studied. The idea is to open the square in the integrand to obtain

$$
\begin{aligned}
\int |V\psi|^2 &= \int |V\psi_1|^2 + \int |V|^2 \psi_1 \bar{\psi}_2 + \int |V|^2 \bar{\psi}_1 \psi_2 + \int |V\psi_2|^2\\
&= I_{11} + I_{12} + \overline{I_{12}} + I_{22}.
\end{aligned}
$$

We will compute these integrals in the next sections. The integral I_{12} is asymptotically real.

6.1.5 The main terms

The evaluations of the main terms coming from integrals I_{11}, I_{12} and I_{22} are now stated as theorems.

Theorem 6.1.5 (Conrey, [Con89])**.** *Suppose $\theta_1 < 1/2$. Then*

$$\int_{-\infty}^{\infty} w(t)|V\psi_1(\sigma_0 + it)|^2 dt \sim c_{11}(P_1, Q, R, \theta_1)\widehat{w}(0) + O(T/L)$$

uniformly for $R \ll 1$, *where*

$$c_{11}(P_1, Q, R, \theta_1) = 1 + \frac{1}{\theta_1}\int_0^1\int_0^1 e^{2Rv}\left(\frac{d}{dx}e^{R\theta x}P_1(x+u)Q(v+\theta x)|_{x=0}\right)^2 dudv. \tag{6.6}$$

Let $(\ell)_k = \ell(\ell-1)\cdot \ldots \cdot (\ell-k+1)$ denote the Pochhammer symbol.

Theorem 6.1.6. *Suppose* $\theta_1 < 1/2 - \varepsilon$ *and* $\theta_2 < 1/2 - \varepsilon$. *Then*

$$\int_{-\infty}^{\infty} w(t) V\psi_1\overline{\psi_2}(\sigma_0 + it)dt \sim c_{12}(P_1, P_\ell, Q, R, \theta_1, \theta_2)\widehat{w}(0) + O(T/L),$$

uniformly for $R \ll 1$, *where*

$$\begin{aligned}
c_{12}(P_1, P_\ell, Q, R, \theta_1, \theta_2) &= \sum_{\ell=2}^{K} \frac{(-1)^\ell}{(\ell-1)!}\int_0^1 (1-u)^{\ell-1}P_1(u)P_\ell(u)du \\
&\quad - \frac{\theta_1-\theta_2}{\theta_1}\sum_{\ell=2}^{K}\frac{(-1)^\ell}{\ell!}\int_0^1 (1-u)^{\ell-1}P_1'\left(1-(1-u)\frac{\theta_2}{\theta_1}\right)P_\ell(u)du \\
&\quad + \frac{1}{\theta_1}\sum_{\ell=2}^{K}\frac{(-1)^\ell}{\ell!}\frac{d^2}{dxdy}\Bigg[e^{R(\theta_1 x+\theta_2 y)} \\
&\quad \times \int_0^1\int_0^1 e^{2Rv}(1-u)^\ell P_1\left(x+1-(1-u)\frac{\theta_2}{\theta_1}\right)P_\ell(y+u) \\
&\quad \times Q(\theta_2 y+v)Q(\theta_1 x+v)dudv\Bigg|_{x=y=0}\Bigg].
\end{aligned} \tag{6.7}$$

Theorem 6.1.7. *Suppose* $\theta_2 < 1/2$. *Then*

$$\int_{-\infty}^{\infty} w(t)|V\psi_2(\sigma_0+it)|^2 dt \sim c_{22}(P_\ell, Q, R, \theta_2)\widehat{w}(0) + O(T/L),$$

uniformly for $R \ll 1$, *where*

$$\begin{aligned}
&c_{22}(P_\ell, Q, R, \theta_2) \\
&= \sum_{\ell_1=2}^{K}\sum_{\ell_2=2}^{K}\sum_{k=0}^{\min(\ell_1,\ell_2)}(-1)^{\ell_1+\ell_2-2k}\binom{\ell_1}{k}(\ell_2)_k \\
&\quad \times \frac{2^{\ell_1+\ell_2-2k}}{(\ell_1+\ell_2-1)!}\int_0^1 (1-u)^{\ell_1+\ell_2-1}P_{\ell_1}(u)P_{\ell_2}(u)du \\
&\quad + \frac{1}{\theta_2}\sum_{\ell_1=2}^{K}\sum_{\ell_2=2}^{K}\sum_{k=0}^{\min(\ell_1,\ell_2)}(-1)^{\ell_1+\ell_2-2k}\binom{\ell_1}{k}(\ell_2)_k\frac{2^{\ell_1+\ell_2-2k}}{(\ell_1+\ell_2)!}\frac{d^2}{dxdy}\Bigg[e^{R\theta_2(x+y)} \\
&\quad \times \int_0^1\int_0^1 e^{2Rv}(1-u)^{\ell_1+\ell_2}P_{\ell_1}(x+u)P_{\ell_2}(y+u)Q(v+\theta_2 x)Q(v+\theta_2 y)dudv\Bigg|_{x=y=0}\Bigg].
\end{aligned} \tag{6.8}$$

Remark 6.1.2. *The result of Conrey, namely Theorem 6.1.5, can be extended to* $\theta_1 < 4/7 - \varepsilon$. *Then Theorem 6.1.6 has to be re-stated with* $\theta_1 < 4/7 - \varepsilon$ *and* $\theta_2 < 3/7 - \varepsilon$. *In Theorem 6.1.7 we may keep* $\theta_2 < 1/2 - \varepsilon$; *however, for the final computation of* κ *we must take* $\min(3/7, 1/2) = 3/7$.

Remark 6.1.3. *Note that in [Fen12], c_{11}, c_{12} and c_{22} are all mixed into one single theorem and it is not immediately clear how to separate each individual c-term.*

The smoothing argument is helpful because we can easily deduce Theorem 6.1.3 from Theorem 6.1.4 and so on. By having chosen $w(t)$ to satisfy conditions (a), (b) and (c) in page 113 and in addition to being an upper bound for the characteristic function of the interval $[T/2, T]$, and with support $[T/2 - \Delta, T + \Delta]$, we obtain

$$\int_{T/2}^{T} |V\psi(\sigma_0 + it)|^2 dt \le c(P_1, P_\ell, Q, R, \theta_1, \theta_2)\widehat{w}(0) + O(T/L).$$

Note that $\widehat{w}(0) = T/2 + O(T/L)$. We similarly obtain a lower bound. Summing over dyadic segments gives the full result.

6.1.6 The shift parameters α and β

Rather than working directly with $V(s)$, we shall instead consider the following three general shifted integrals

$$\begin{aligned}
I_{11}(\alpha,\beta) &= \int_{-\infty}^{\infty} w(t)\zeta(\tfrac{1}{2}+\alpha+it)\zeta(\tfrac{1}{2}+\beta-it)\overline{\psi_1}\psi_1(\sigma_0+it)dt,\\
I_{12}(\alpha,\beta) &= \int_{-\infty}^{\infty} w(t)\zeta(\tfrac{1}{2}+\alpha+it)\zeta(\tfrac{1}{2}+\beta-it)\overline{\psi_1}\psi_2(\sigma_0+it)dt,\\
I_{22}(\alpha,\beta) &= \int_{-\infty}^{\infty} w(t)\zeta(\tfrac{1}{2}+\alpha+it)\zeta(\tfrac{1}{2}+\beta-it)\overline{\psi_2}\psi_2(\sigma_0+it)dt.
\end{aligned}$$

The computation is now reduced to proving the following three lemmas.

Lemma 6.1.1. *We have*

$$I_{11} = c_{11}(\alpha,\beta)\widehat{w}(0) + O(T/L),$$

uniformly for $\alpha, \beta \ll L^{-1}$, where

$$c_{11}(\alpha,\beta) = 1 + \frac{1}{\theta_1}\frac{d^2}{dxdy}\left[y_1^{-\beta x-\alpha y}\int_0^1\int_0^1 T^{-v(\alpha+\beta)}P_1(x+u)P_1(y+u)dudv\bigg|_{x=y=0}\right]. \tag{6.9}$$

Lemma 6.1.2. *We have*

$$I_{12} = c_{12}(\alpha,\beta)\widehat{w}(0) + O(T/L),$$

uniformly for $\alpha, \beta \ll L^{-1}$, where

$$\begin{aligned}
c_{12}(\alpha,\beta) =& \sum_{\ell=2}^{K}\frac{(-1)^\ell}{(\ell-1)!}\int_0^1 (1-u)^{\ell-1}P_1(u)P_\ell(u)du\\
&- \frac{\theta_1-\theta_2}{\theta_1}\sum_{\ell=2}^{K}\frac{(-1)^\ell}{\ell!}\int_0^1 (1-u)^\ell P_1'\left(1-(1-u)\frac{\theta_2}{\theta_1}\right)P_\ell(u)du\\
&+ \frac{1}{\theta_1}\sum_{\ell=2}^{K}\frac{(-1)^\ell}{\ell!}\frac{d^2}{dxdy}\Bigg[y_1^{-\beta x}y_2^{-\alpha y}
\end{aligned}$$

$$\times \int_0^1 \int_0^1 T^{-v(\alpha+\beta)}(1-u)^{\ell} P_1\left(x+1-(1-u)\frac{\theta_2}{\theta_1}\right) P_\ell(y+u)dudv\Big|_{x=y=0}\Bigg]. \tag{6.10}$$

Lemma 6.1.3. *We have*

$$I_{22} = c_{22}(\alpha,\beta)\widehat{w}(0) + O(T/L),$$

uniformly for $\alpha, \beta \ll L^{-1}$, where

$$\begin{aligned} & c_{22}(\alpha,\beta) \\ &= \sum_{\ell_1=2}^{K}\sum_{\ell_2=2}^{K}\sum_{k=0}^{\min(\ell_1,\ell_2)} (-1)^{\ell_1+\ell_2-2k}\binom{\ell_1}{k}(\ell_2)_k \\ &\times \frac{2^{\ell_1+\ell_2-2k}}{(\ell_1+\ell_2-1)!}\int_0^1 (1-u)^{\ell_1+\ell_2-1}P_{\ell_1}(u)P_{\ell_2}(u)du \\ &+ \frac{1}{\theta_2}\sum_{\ell_1=2}^{K}\sum_{\ell_2=2}^{K}\sum_{k=0}^{\min(\ell_1,\ell_2)} \binom{\ell_1}{k}(\ell_2)_k(-1)^{\ell_1+\ell_2-2k}\frac{2^{\ell_1+\ell_2-2k}}{(\ell_1+\ell_2)!} \\ &\times \frac{d^2}{dxdy}\left[y_2^{-\beta x-\alpha y}\int_0^1\int_0^1 T^{-v(\alpha+\beta)}(1-u)^{\ell_1+\ell_2}P_{\ell_1}(x+u)P_{\ell_2}(y+u)dudv\Big|_{x=y=0}\right]. \end{aligned} \tag{6.11}$$

To prove Theorems 6.1.5, 6.1.6 and 6.1.7 we use the following technique. Let $I_\star$ denote either of the integrals in questions, and note that

$$I_\star = Q\left(-\frac{1}{\log T}\frac{d}{d\alpha}\right)Q\left(-\frac{1}{\log T}\frac{d}{d\beta}\right)I_*(\alpha,\beta)\Big|_{\alpha=\beta=R/L}.$$

Since $I_\star(\alpha,\beta)$ and $c_\star(\alpha,\beta)$ are holomorphic with respect to α,β small, the derivatives appearing in the equation above can be obtained as integrals of radii $\asymp L^{-1}$ around the points $-R/L$, using Cauchy's integral formula. Since the error terms hold uniformly on these contours, the same error terms that hold for $I_\star(\alpha,\beta)$ also hold for $I_\star$. That the above differential operator on $c_\star(\alpha,\beta)$ does indeed give $c_\star$ follows from

$$Q\left(\frac{-1}{\log T}\frac{d}{d\alpha}X^{-\alpha}\right) = Q\left(\frac{\log X}{\log T}\right)X^{-\alpha}.$$

Note that from the above equation we obtain

$$\begin{aligned} & Q\left(\frac{-1}{\log T}\frac{d}{d\alpha}\right)Q\left(\frac{-1}{\log T}\frac{d}{d\beta}\right)y_1^{-\beta x}y_2^{-\alpha y}T^{-v(\alpha+\beta)} \\ &= Q\left(\frac{\log y_2^y T^v}{\log T}\right)Q\left(\frac{\log y_1^x T^v}{\log T}\right)y_1^{-\beta x}y_2^{-\alpha y}T^{-v(\alpha+\beta)} \\ &= Q(\theta_2 y+v)Q(\theta_1 x+v)y_1^{-\beta x}y_2^{-\alpha y}T^{-v(\alpha+\beta)}, \end{aligned}$$

as well as

$$\begin{aligned} & Q\left(\frac{-1}{\log T}\frac{d}{d\alpha}\right)Q\left(\frac{-1}{\log T}\frac{d}{d\beta}\right)y_2^{-\beta x-\alpha y}T^{-v(\alpha+\beta)} \\ &= Q\left(\frac{\log y_2^y T^v}{\log T}\right)(y_2^y T^v)^{-\alpha}Q\left(\frac{\log y_2^x T^v}{\log T}\right)(y_2^x T^v)^{-\beta} \end{aligned}$$

$$= Q(\theta_2 y + v)Q(\theta_2 x + v)y_2^{-\beta x - \alpha y}T^{-v(\alpha+\beta)}.$$

Hence using the differential operators $Q((-1/\log T)d/d\alpha)$ and $Q((-1/\log T)d/d\beta)$ on the last line of $c_{12}(\alpha,\beta)$ we get

$$\frac{d^2}{dxdy}\left[y_1^{-\beta x}y_2^{-\alpha y}\int_0^1\int_0^1 T^{-v(\alpha+\beta)}(1-u)^{\ell}P_1\left(x+1-(1-u)\frac{\theta_2}{\theta_1}\right)\right.$$
$$\left.P_\ell(y+u)Q(\theta_2 y+v)Q(\theta_1 x+v)dudv\bigg|_{x=y=0}\right].$$

Theorem 6.1.6 then follows by setting $\alpha=\beta=-R/L$ and using $T^{z/L}=T^{z/\log T}=e^z$. Similarly, when we use the differential operators $Q((-1/\log T)d/d\alpha)$ and $Q((-1/\log T)d/d\beta)$ on the last line of $c_{22}(\alpha,\beta)$ it becomes

$$\frac{d^2}{dxdy}\left[e^{R\theta_2(x+y)}\int_0^1\int_0^1 e^{2Rv}(1-u)^{\ell_1+\ell_2}P_{\ell_1}(x+u)P_{\ell_2}(y+u)\right.$$
$$\left.\times Q(v+\theta_2 x)Q(v+\theta_2 y)dudv\bigg|_{x=y=0}\right].$$

The same substitutions yield Theorem 6.1.7.

6.2 Preliminary results

6.2.1 Results from complex analysis

The following results are needed throughout this chapter.

Lemma 6.2.1. *Suppose that $w(t)$ satisfies conditions* (a), (b) *and* (c) *in page 113, and that h, k are positive integers with $hk \le T^{2\theta}$ with $\theta < 1/2$, and $\alpha,\beta \ll L^{-1}$. Moreover, set*

$$g_{\alpha,\beta}(s,t) = \pi^{-s}\frac{\Gamma(\frac{1}{2}(\frac{1}{2}+\alpha+s+it))\Gamma(\frac{1}{2}(\frac{1}{2}+\beta+s-it))}{\Gamma(\frac{1}{2}(\frac{1}{2}+\alpha+it))\Gamma(\frac{1}{2}(\frac{1}{2}+\beta-it))},$$

as well as

$$X_{\alpha,\beta,t} = \pi^{\alpha+\beta}\frac{\Gamma(\frac{1}{2}(\frac{1}{2}-\alpha-it))\Gamma(\frac{1}{2}(\frac{1}{2}-\beta+it))}{\Gamma(\frac{1}{2}(\frac{1}{2}+\alpha+it))\Gamma(\frac{1}{2}(\frac{1}{2}+\beta-it))}.$$

Then one has

$$\begin{aligned}&\int_{-\infty}^{\infty} w(t)\left(\frac{h}{k}\right)^{-it}\zeta(\tfrac{1}{2}+\alpha+it)\zeta(\tfrac{1}{2}+\beta-it)dt\\ &= \sum_{hm=kn}\frac{1}{m^{1/2+\alpha}n^{1/2+\beta}}\int_{-\infty}^{\infty}V_{\alpha,\beta}(mn,t)w(t)dt\\ &\quad+\sum_{hm=kn}\frac{1}{m^{1/2-\beta}n^{1/2-\alpha}}\int_{-\infty}^{\infty}V_{-\beta,-\alpha}(mn,t)X_{\alpha,\beta,t}w(t)dt + O_A(T^{-A}),\end{aligned}$$

where

$$V_{\alpha,\beta}(x,t) = \frac{1}{2\pi i}\int_{(1)}\frac{G(s)}{s}g_{\alpha,\beta}(s,t)x^{-s}ds,\ G(s)=e^{s^2}p(s) \text{ and } p(s)=\frac{(\alpha+\beta)^2-(2s)^2}{(\alpha+\beta)^2}.$$

Proof. See Lemma 5 of [You10]. They key point is that non-diagonal terms $hm \neq kn$ can safely be absorbed in the error terms. □

Lemma 6.2.2. *Suppose $0 < \delta \asymp L^{-1}$ and $\beta \ll L^{-1}$. For some $\nu \asymp (\log\log y)^{-1}$ we have*

$$\begin{aligned}\Upsilon &:= \frac{1}{2\pi i}\int_{(\delta)} \frac{1}{\zeta(1+\beta+u)}\left(\frac{\zeta'}{\zeta}(1+\beta+u)\right)^{\ell-r}\left(\frac{y}{n}\right)^{u}\frac{du}{u^{j+1}} \\ &= (-1)^{\ell-r}\frac{1}{2\pi i}\oint (\beta+u)^{1-\ell+r}\left(\frac{y}{n}\right)^{u}\frac{du}{u^{j+1}} + O(L^{\ell-r-2+j}) + O\left(\left(\frac{y}{n}\right)^{-\nu}L^{\varepsilon}\right),\end{aligned}$$

where the contour is a circle of radius one enclosing the origin and $-\beta$.

Proof. This follows a similar procedure to Lemma 6.1 of [BCY11] where the zero-free region of ζ is used. Let $Y = o(T)$ be a large parameter to be chosen later. By Cauchy's theorem, L_1 is equal to the sum of residues at $u = 0$ and $u = -\beta$ plus integrals over the line segments $\gamma_1 = \{s = it : t \in \mathbb{R}, |t| \geq Y\}$, $\gamma_2 = \{s = \sigma \pm iY : -c/\log Y \leq \sigma \leq 0\}$, and $\gamma_3 = \{s = -c/\log Y + it : |t| \leq Y\}$, where c is some fixed positive constant such that $\zeta(1+\beta+u)$ has no zeros in the region on the right-hand side of the contour determined by the γ_i's. Another requirement on c is that the estimate (see [Tit86, Theorem 3.11])

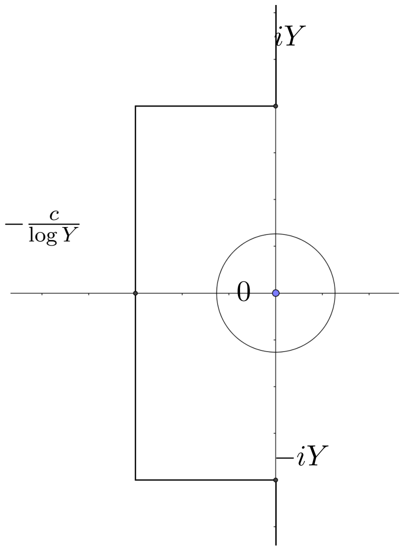

FIGURE 6.1: Curve γ in the proof of Lemma 6.2.2.

$1/\zeta(\sigma+it) \ll \log(2+|t|)$ holds in this region and $\zeta'/\zeta(\sigma+it) \ll \log(4+|t|)$ (see [MV07, Theorem 6.7]). Then, one has

$$\int_{\gamma_1} \ll \int_Y^\infty \frac{\log(t)^{1+\ell-r}}{t^{j+1}}\,dt \ll \frac{\log(Y)^{1+\ell-r}}{Y^j},$$

since $j \geq 3$. Moreover, since $n \leq y$,

$$\int_{\gamma_2} \ll \int_{-c/\log Y}^{0} \log(Y)^{1+\ell-r}\left(\frac{y}{n}\right)^x \frac{1}{Y^{j+1}}\,dx \ll \frac{\log(Y)^{\ell-r}}{Y^{j+1}},$$

and finally

$$\int_{\gamma_3} \ll \int_{-Y}^{Y} \log(4+|t|)^{\ell-r+1} \frac{(y/n)^{-c/\log Y}}{c^2/\log^2 Y + t^2}\, dt \ll \log(Y)^{\ell-r+j}(y/n)^{-c/\log Y}.$$

Appropriately choosing $Y \asymp (\log y)$ gives an error of size $O((\log\log y)^{\ell-r+j}) = O(\log y)$.

The next step is to sum the residues. This sum can now be expressed as

$$\frac{1}{2\pi i}\oint \frac{1}{\zeta(1+\beta+u)}\left(\frac{\zeta'}{\zeta}(1+\beta+u)\right)^{\ell-r}\left(\frac{y}{n}\right)^{u}\frac{du}{u^{j+1}},$$

where the contour is now a small circle of radius $\asymp 1/L$ around the origin and $-\beta \in \Omega$. Since the radius of the circle is tending to zero, we can use the Laurent expansions

$$\frac{1}{\zeta(s)} = s - 1 + O((s-1)^2) \quad \text{and} \quad \frac{\zeta'}{\zeta}(s) = -\frac{1}{s-1} + \gamma + O(|s-1|),$$

to finally obtain

$$\begin{aligned} &\frac{1}{2\pi i}\oint \frac{1}{\zeta(1+\beta+u)}\left(\frac{\zeta'}{\zeta}(1+\beta+u)\right)^{\ell-r}\left(\frac{y}{n}\right)^{u}\frac{du}{u^{j+1}} \\ &= \frac{1}{2\pi i}\oint (\beta+u+O(u^2))\left(\frac{-1}{\beta+u}+O(1)\right)^{\ell-r}\left(\frac{y}{n}\right)^{u}\frac{du}{u^{j+1}}. \end{aligned}$$

Using the binomial theorem and a direct estimate, we conclude that the above term is equal to

$$(-1)^{\ell-r}\oint (\beta+u)^{1-\ell+r}\left(\frac{y}{n}\right)^{u}\frac{du}{u^{j+1}} + O(L^{j+\ell-r-2}), \tag{6.12}$$

which is the desired main term of the lemma. □

The integral in (6.12) can be computed exactly. To do this, note that for any integer $k \geq 1$, one has

$$q^u(\beta+u)^k = \frac{d^k}{dy^k} e^{\beta y}(e^y q)^u\Big|_{y=0}.$$

Hence, one arrives at

$$\begin{aligned} \Upsilon &= (-1)^{\ell-r}\frac{1}{2\pi i}\oint \frac{d^{1-\ell+r}}{dy^{1-\ell+r}} e^{\beta y}(e^y q)^u\Big|_{y=0}\frac{du}{u^{j+1}} \\ &= (-1)^{\ell-r}\frac{d^{1-\ell+r}}{dy^{1-\ell+r}} e^{\beta y}\frac{1}{2\pi i}\oint (e^y q)^u|_{y=0}\frac{du}{u^{j+1}} \\ &= \frac{(-1)^{\ell-r}}{j!}\frac{d^{1-\ell+r}}{dy^{1-\ell+r}} e^{\beta y}\left(y+\log\frac{y}{n}\right)^{j}\Bigg|_{y=0}, \end{aligned} \tag{6.13}$$

by Cauchy's integral theorem and where we temporarily set $q = y/n$.

6.2.2 Combinatorial results

When computing the crossterm of ψ_2 and ψ_2 the following result will be needed.

Lemma 6.2.3. *For h_1 and h_2 square-free, we have*

$$\begin{aligned}\mathcal{Q}(\ell_1,\ell_2) &:= \sum_{p_1p_2\cdots p_{\ell_1}|h_1} \log p_1 \log p_2 \cdots \log p_{\ell_1} \sum_{q_1q_2\cdots q_{\ell_2}|h_2} \log q_1 \log q_2 \cdots \log q_{\ell_2} \\ &= \sum_{k=0}^{\min\{\ell_1,\ell_2\}} k! \binom{\ell_1}{k}\binom{\ell_2}{k} \sum_{\substack{p_1p_2\cdots p_k q_1q_2\cdots q_{\ell_1-k} r_1r_2\cdots r_{\ell_2-k}|h_1h_2 \\ p_1p_2\cdots p_k|\gcd(h_1,h_2) \\ q_1\cdots q_{\ell_1-k}|h_1 \\ r_1\cdots r_{\ell_2-k}|h_2}} \\ &\quad \times \left(\prod_{f=1}^{k} \log^2 p_f\right)\left(\prod_{f=1}^{\ell_1-k} \log q_f\right)\left(\prod_{f=1}^{\ell_2-k} \log r_f\right),\end{aligned}$$

Here the p's, the q's and the r's are all distinct primes.

Proof. We may write

$$\begin{aligned}\mathcal{Q}(\ell_1,\ell_2) &= \sum_{\substack{p_1p_2\cdots p_{\ell_1}|h_1 \\ p_a\neq p_b}} \log p_1 \log p_2 \cdots \log p_{\ell_1} \sum_{\substack{q_1q_2\cdots q_{\ell_2}|h_2 \\ q_a\neq q_b}} \log q_1 \log q_2 \cdots \log q_{\ell_2} \\ &= \ell_1!\ell_2! \sum_{\substack{p_1p_2\cdots p_{\ell_1}|h_1 \\ p_1<p_2<\cdots<p_{\ell_1}}} \log p_1 \log p_2 \cdots \log p_{\ell_1} \sum_{\substack{q_1q_2\cdots q_{\ell_2}|h_2 \\ q_1<q_2<\cdots<q_{\ell_2}}} \log q_1 \log q_2 \cdots \log q_{\ell_2} \\ &= \ell_1!\ell_2! \sum_{k=0}^{\min\{\ell_1,\ell_2\}} \sum_{\substack{p_1p_2\cdots p_k q_1q_2\cdots q_{\ell_1-k} r_1r_2\cdots r_{\ell_2-k}|h_1h_2 \\ p_1p_2\cdots p_k|\gcd(h_1,h_2),\, q_1\cdots q_{\ell_1-k}|h_1,\, r_1\cdots r_{\ell_2-k}|h_2 \\ p_1<p_2<\cdots<p_k,\, q_1<q_2<\cdots<q_{\ell_1-k},\, r_1<r_2<\cdots<r_{\ell_2-k}}} \\ &\quad \times \left(\log^2 p_1 \cdots \log^2 p_k\right)\left(\log q_1 \cdots \log q_{\ell_1-k}\right)\left(\log r_1 \cdots \log r_{\ell_2-k}\right) \\ &= \sum_{k=0}^{\min\{\ell_1,\ell_2\}} \frac{\ell_1!\ell_2!}{k!(\ell_1-k)!(\ell_2-k)!} \sum_{\substack{p_1p_2\cdots p_k q_1q_2\cdots q_{\ell_1-k} r_1r_2\cdots r_{\ell_2-k}|h_1h_2 \\ p_1p_2\cdots p_k|\gcd(h_1,h_2),\, q_1\cdots q_{\ell_1-k}|h_1,\, r_1\cdots r_{\ell_2-k}|h_2}} \\ &\quad \times \left(\log^2 p_1 \cdots \log^2 p_k\right)\left(\log q_1 \cdots \log q_{\ell_1-k}\right)\left(\log r_1 \cdots \log r_{\ell_2-k}\right).\end{aligned}$$

Using the definition of the binomial coefficient completes the proof. □

6.2.3 Generalized von Mangoldt functions and Euler-MacLaurin summations

Recall that for a positive integer k, the generalized von Mangoldt function $\Lambda_k(n)$ is defined [Ivi75] by the Dirichlet convolution

$$\Lambda_k(n) = (\mu * \log^k)(n),$$

so that $\Lambda_1(n) = \Lambda(n)$. The generating series is

$$\sum_{n=1}^{\infty} \frac{\Lambda_k(n)}{n^s} = (-1)^k \frac{\zeta^{(k)}}{\zeta}(s),$$

for $\mathrm{Re}(s) > 1$. By looking at

$$\frac{d}{ds}\left(\frac{\zeta^{(k)}}{\zeta}(s)\right) = \frac{\zeta^{(k+1)}}{\zeta}(s) - \frac{\zeta'}{\zeta}(s)\frac{\zeta^{(k)}}{\zeta}(s)$$

for $\mathrm{Re}(s) > 1$, we see that

$$\Lambda_{k+1}(n) = \Lambda_k(n)\log(n) + (\Lambda * \Lambda_k)(n),$$

and in particular for $k = 1$

$$\Lambda_2(n) = \Lambda(n)\log(n) + (\Lambda * \Lambda)(n).$$

Lemma 6.2.4. *We have for smooth functions F and G in the interval $[0,1]$, $3 \le z \le x$, and $|s| \le (\log x)^{-1}$,*

$$\begin{aligned}&\sum_{n\le z}\frac{\Lambda(n)\log n}{n^{1+s}}F\left(\frac{\log(x/n)}{\log x}\right)H\left(\frac{\log(z/n)}{\log z}\right)\\ &= \frac{\log^2 z}{z^s}\int_0^1(1-u)F\left(1-(1-u)\frac{\log z}{\log x}\right)H(u)\,z^{us}du\\ &\quad + O(\log z)\end{aligned}$$

as $z \to \infty$.

Proof. Start by setting

$$\Psi(z,x) := \sum_{n\le z}\frac{\Lambda(n)\log n}{n^{1+s}}F\left(\frac{\log(x/n)}{\log x}\right)H\left(\frac{\log(z/n)}{\log z}\right) \quad \text{and} \quad \psi(x) := \sum_{n\le x}\Lambda(n).$$

By applying the Abel summation formula, we obtain

$$\begin{aligned}&\Psi(z,x)\\ &= \psi(z)\frac{\log z}{z^{1+s}}F\left(\frac{\log(x/z)}{\log x}\right)H(0)\\ &\quad - \int_1^z \psi(u)\frac{d}{du}\left(\frac{\log u}{u^{1+s}}F\left(\frac{\log(x/u)}{\log x}\right)H\left(\frac{\log(z/u)}{\log z}\right)\right)du\\ &= -\int_1^z \psi(u)\frac{d}{du}\left(\frac{\log u}{u^{1+s}}F\left(\frac{\log(x/u)}{\log x}\right)H\left(\frac{\log(z/u)}{\log z}\right)\right)du + O(\log z)\\ &= -\int_1^z \psi(u)\frac{1-(1+s)\log u}{u^{2+s}}F\left(\frac{\log(x/u)}{\log x}\right)H\left(\frac{\log(z/u)}{\log z}\right)du\\ &\quad - \int_1^z \psi(u)\frac{\log u}{u^{1+s}}\left(\frac{d}{du}F\left(\frac{\log(x/u)}{\log x}\right)\right)H\left(\frac{\log(z/u)}{\log z}\right)du\\ &\quad - \int_1^z \psi(u)\frac{\log u}{u^{1+s}}F\left(\frac{\log(x/u)}{\log x}\right)\left(\frac{d}{du}H\left(\frac{\log(z/u)}{\log z}\right)\right)du + O(\log z)\\ &= \frac{\log^2 z(1+s)}{z^s}\int_0^1 \psi(z^{1-b})(1-b)F\left(1-(1-b)\frac{\log z}{\log x}\right)H(b)\,z^{bs+b-1}db\\ &\quad + O\left(\log z\int_1^z \psi(u)\frac{1}{u^{2+|s|}}du\right)\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{\log x}\int_1^z \psi(u)\frac{\log u}{u^{2+s}}F'\left(\frac{\log(x/u)}{\log x}\right)H\left(\frac{\log(z/u)}{\log z}\right)du\\
&+\frac{1}{\log z}\int_1^z \psi(u)\frac{\log u}{u^{2+s}}F\left(\frac{\log(x/u)}{\log x}\right)H'\left(\frac{\log(z/u)}{\log z}\right)du+O\left(\log z\right)\\
=&\frac{\log^2 z(1+s)}{z^s}\int_0^1 \psi(z^{1-b})(1-b)F\left(1-(1-b)\frac{\log z}{\log x}\right)H\left(b\right)z^{bs+b-1}db+O\left(\log z\right)\\
=&\frac{\log^2 z}{z^s}\int_0^1 \psi(z^{1-b})(1-b)F\left(1-(1-b)\frac{\log z}{\log x}\right)H\left(b\right)z^{bs+b-1}db\\
&+O\left(\log z\int_0^1 \psi(z^{1-b})(1-b)z^{bs+b-1}db\right)+O\left(\log z\right)\\
=&\frac{\log^2 z}{z^s}\int_0^1 (1-b)F\left(1-(1-b)\frac{\log z}{\log x}\right)H\left(b\right)z^{bs}db+O\left(\log z\int_0^1(1-b)z^{bs}db\right)\\
&+O\left(\log z\right)\\
=&\frac{\log^2 z}{z^s}\int_0^1 (1-b)F\left(1-(1-b)\frac{\log z}{\log x}\right)H\left(b\right)z^{bs}db+O\left(\log z\right),
\end{aligned}
$$

since $\psi(x) = x + O(x\exp(-c\sqrt{\log x}))$ for $c > 1$ by the prime number theorem with remainder, see e.g. [Tit86]. □

Lemma 6.2.5. *We have for smooth functions F and G in the interval $[0,1]$, $3 \leq z \leq x$, and $|s| \leq (\log x)^{-1}$*

$$
\begin{aligned}
\sum_{n\leqslant z}\frac{(d_k*\Lambda^{*l})}{n^{1+s}}F\left(\frac{\log x/n}{\log x}\right)H\left(\frac{\log z/n}{\log z}\right)&\\
=\frac{(\log z)^{k+l}}{(k+l-1)!z^s}\int_0^1 (1-u)^{k+l-1}F\left(1-(1-u)\frac{\log z}{\log x}\right)H(u)z^{us}du&\\
+O((\log 3z)^{k+l-1}),&
\end{aligned}
\tag{6.14}
$$

where $d_k(n)$ denotes the number of ways an integer n can be written as a product of $k \geq 2$ fixed factors. Note that $d_1(n) = 1$ and $d_2(n) = d(n)$, the number of divisors of n.

Proof. This is Lemma 3.6 of [RRZ16]. □

Lemma 6.2.6. *We have for smooth functions F and G in the interval $[0,1]$, $3 \leq z \leq x$, and $|s| \leq (\log x)^{-1}$*

$$
\begin{aligned}
\sum_{n\leq z}\frac{(1*\Lambda^{*a}*\Lambda\log)(n)}{n^{1+s}}F\left(\frac{\log(x/n)}{\log x}\right)H\left(\frac{\log(z/n)}{\log z}\right)&\\
=\frac{\log^{3+a} z}{(a+2)!z^s}\int_0^1 (1-u)^{a+2}F\left(1-(1-u)\frac{\log z}{\log x}\right)H\left(u\right)z^{us}du&\\
+O(\log^{a+2} z).&
\end{aligned}
$$

Proof. The proof is the same as in the beginning of the proof of Lemma 6.2.5 but instead we use Lemma 6.2.4. □

Lemma 6.2.7. *We have for smooth functions F and G in the interval $[0,1]$, $3 \leq z \leq x$, and $|s| \leq (\log x)^{-1}$*

$$\sum_{n\leq z}\frac{(1*\Lambda^{*a}*\Lambda_2^{*b})(n)}{n^{1+s}}F\left(\frac{\log(x/n)}{\log x}\right)H\left(\frac{\log(z/n)}{\log z}\right)$$
$$=2^b\frac{\log^{1+a+2b}z}{(a+2b)!z^s}\int_0^1(1-u)^{a+2b}F\left(1-(1-u)\frac{\log z}{\log x}\right)H\left(u\right)z^{us}du+O(\log^{a+2b}z).$$

Proof. Follows by induction on b and by using Lemma 6.2.6 and $\Lambda_2(n) = (\Lambda\log)(n) + (\Lambda*\Lambda)(n)$. □

6.3 Evaluation of the shifted mean value integrals $I_\star(\alpha,\beta)$

6.3.1 The mean value integral $I_{11}(\alpha,\beta)$

Although this was already explained in [You10], the mean value integral $I_{22}(\alpha,\beta)$ builds up from $I_{12}(\alpha,\beta)$ which in turn is a refinement of $I_{11}(\alpha,\beta)$. Therefore, it repays a careful analysis to go over the main points of the evaluation of $I_{11}(\alpha,\beta)$ briefly. We start by inserting the definition of the mollifier ψ_1 in I_{11} so that

$$\begin{aligned}
I_{11}(\alpha,\beta) &= \int_{-\infty}^{\infty}w(t)\zeta(\tfrac12+\alpha+it)\zeta(\tfrac12+\beta-it)\psi_1\overline{\psi_1}(\sigma_0+it)dt\\
&=\int_{-\infty}^{\infty}w(t)\zeta(\tfrac12+\alpha+it)\zeta(\tfrac12+\beta-it)\\
&\quad\times\sum_{h\leqslant y_1}\frac{\mu(h)h^{-1/2}}{h^{it}}P_1\left(\frac{\log y_1/h}{\log y_1}\right)\sum_{k\leqslant y_1}\frac{\mu(k)k^{-1/2}}{k^{-it}}P_1\left(\frac{\log y_1/k}{\log y_1}\right)dt\\
&=\sum_{h\leqslant y_1}\sum_{k\leqslant y_1}\frac{\mu(h)\mu(k)}{(hk)^{1/2}}P_1\left(\frac{\log y_1/h}{\log y_1}\right)P_1\left(\frac{\log y_1/k}{\log y_1}\right)\\
&\quad\times\int_{-\infty}^{\infty}w(t)\left(\frac{h}{k}\right)^{-it}\zeta(\tfrac12+\alpha+it)\zeta(\tfrac12+\beta-it)dt.
\end{aligned}$$

According to Lemma 6.2.1, we write $I_{11}(\alpha,\beta) = I'_{11}(\alpha,\beta)+I''_{11}(\alpha,\beta)$, where I'_{11} is given by

$$\begin{aligned}
I'_{11}(\alpha,\beta) &= \sum_{h\leqslant y_1}\sum_{k\leqslant y_1}\frac{\mu(h)\mu(k)}{(hk)^{1/2}}P_1\left(\frac{\log y_1/h}{\log y_1}\right)P_1\left(\frac{\log y_1/k}{\log y_1}\right)\\
&\quad\times\sum_{hm=kn}\frac{1}{m^{1/2+\alpha}n^{1/2+\beta}}\int_{-\infty}^{\infty}V_{\alpha,\beta}(mn,t)w(t)dt.
\end{aligned}\tag{6.15}$$

Notice that $I''_{11}(\alpha,\beta)$ is obtained by replacing α with $-\beta$, β with $-\alpha$ and multiplying inside the integrand by $X_{\alpha,\beta,t} = T^{-\alpha,\beta}(1+O(L^{-1}))$. In other words,

$$I_{11}(\alpha,\beta) = I'_{11}(\alpha,\beta)+T^{-\alpha-\beta}I'_{11}(-\beta,-\alpha)+O(T/L).$$

Let us then look at I'_{11} more closely. Using the Mellin representations

$$P_1[h] = \sum_i\frac{a_i i!}{\log^i y_1}\frac{1}{2\pi i}\int_{(1)}\left(\frac{y_1}{h}\right)^s\frac{ds}{s^{i+1}}\quad\text{and}\quad P_1[k] = \sum_j\frac{a_j j!}{\log^j y_1}\frac{1}{2\pi i}\int_{(1)}\left(\frac{y_1}{k}\right)^u\frac{du}{u^{j+1}},$$

we then obtain

$$I'_{11}(\alpha,\beta) = \int_{-\infty}^{\infty} w(t) \sum_{i,j} \frac{a_i i! a_j j!}{\log^{i+j} y_1} \left(\frac{1}{2\pi i}\right)^3 \int_{(1)}\int_{(1)}\int_{(1)} y_1^{s+u} g_{\alpha,\beta}(z,t) \frac{G(z)}{z}$$
$$\times \sum_{hm=kn} \frac{\mu(h)\mu(k)}{h^{1/2+s}k^{1/2+u}m^{1/2+\alpha+z}n^{1/2+\beta+z}} dz \frac{ds}{s^{i+1}} \frac{du}{u^{j+1}} dt.$$

We now evaluate the arithmetical sum $S = \sum_{hm=kn}$ in the integrand. This is done p-adically as follows. We denote by $\nu_p(n)$ the number of times the prime number p appears in n, and without risk of confusion we write $n' = \nu_p(n)$. This means that

$$\begin{aligned} S &= \sum_{hm=kn} \frac{\mu(h)\mu(k)}{h^{1/2+s}k^{1/2+u}m^{1/2+\alpha+z}n^{1/2+\beta+z}} \\ &= \prod_p \sum_{h'+n'=m'+k'} \frac{\mu(p^{h'})\mu(p^{k'})}{(p^{h'})^{1/2+s}(p^{k'})^{1/2+u}(p^{m'})^{1/2+\alpha+z}(p^{n'})^{1/2+\beta+z}} \\ &= \prod_p \left(1 + \frac{1}{p^{1+s+u}} - \frac{1}{p^{1+s+\alpha+z}} - \frac{1}{p^{1+u+\beta+z}} + \frac{1}{p^{1+\alpha+\beta+2z}} + O(p^{-2+\varepsilon})\right) \\ &= \frac{\zeta(1+s+u)\zeta(1+\alpha+\beta+2z)}{\zeta(1+s+\alpha+z)\zeta(1+u+\beta+z)} A_{\alpha,\beta}(s,u,z), \end{aligned}$$

where the arithmetical factor $A_{\alpha,\beta}(s,u,z)$ is given by an absolutely convergent Euler product in some product of half-planes containing the origin. It will important to remark that when $\alpha = \beta = 0$ and $s = u = z$ we have

$$A_{0,0}(z,z,z) = \sum_{hm=kn} \frac{\mu(h)\mu(k)}{h^{1/2+z}k^{1/2+z}m^{1/2+z}n^{1/2+z}} = \sum_{hm=kn} \frac{\mu(h)\mu(k)}{(hkmn)^{1/2+z}} = 1, \tag{6.16}$$

for all z, by the Möbius inversion formula. Inserting this into I'_{11} we get

$$I'_{11}(\alpha,\beta) = \int_{-\infty}^{\infty} w(t) \sum_{i,j} \frac{a_i i! a_j j!}{\log^{i+j} y_1} \left(\frac{1}{2\pi i}\right)^3 \int_{(1)}\int_{(1)}\int_{(1)} y_1^{s+u} g_{\alpha,\beta}(z,t) \frac{G(z)}{z}$$
$$\times \frac{\zeta(1+s+u)\zeta(1+\alpha+\beta+2z)}{\zeta(1+s+\alpha+z)\zeta(1+u+\beta+z)} A_{\alpha,\beta}(s,u,z) dz \frac{ds}{s^{i+1}} \frac{du}{u^{j+1}} dt.$$

Now, we deform the path of integration to $\mathrm{Re}(z) = -\delta + \varepsilon$ where $\delta > 0$ is small and $\mathrm{Re}(s) = \mathrm{Re}(u) = \delta$. By doing this, we pick up a simple pole coming from $1/z$ at $z = 0$ only, since $G(z)$ vanishes at the pole of $\zeta(1+\alpha+\beta+2z)$. The new path of integration contributes an error of the size

$$\sum_{n \leq y_1} \frac{1}{n} \left(1 + \log \frac{y_1}{n}\right)^{-2} \ll 1 \ll L^{i+j-2}.$$

Thus, we end up with

$$I'_{11}(\alpha,\beta) = \int_{-\infty}^{\infty} w(t) \sum_{i,j} \frac{a_i i! a_j j!}{\log^{i+j} y_1} \left(\frac{1}{2\pi i}\right)^2 \int_{(1)}\int_{(1)} \operatorname*{Res}_{z=0} y_1^{s+u} g_{\alpha,\beta}(z,t) \frac{G(z)}{z}$$
$$\times \frac{\zeta(1+s+u)\zeta(1+\alpha+\beta+2z)}{\zeta(1+s+\alpha+z)\zeta(1+u+\beta+z)} A_{\alpha,\beta}(s,u,z) \frac{ds}{s^{i+1}} \frac{du}{u^{j+1}} dt$$

$$= \widehat{w}(0)\zeta(1+\alpha+\beta)\sum_{i,j}\frac{a_i i! a_j j!}{\log^{i+j} y_1} J_{11},$$

where

$$J_{11} = \left(\frac{1}{2\pi i}\right)^2 \int_{(1)}\int_{(1)} y_1^{s+u} \frac{\zeta(1+s+u)}{\zeta(1+s+\alpha)\zeta(1+u+\beta)} A_{\alpha,\beta}(s,u,0)\frac{ds}{s^{i+1}}\frac{du}{u^{j+1}}.$$

Using the Dirichlet series representation for $\zeta(1+s+u)$, we can separate the complex variables s and u. The next step is to use the Laurent expansion

$$\begin{aligned}\frac{A_{\alpha,\beta}(s,u,0)}{\zeta(1+s+\alpha)\zeta(1+u+\beta)} &= (\alpha+s)(\beta+u)A_{0,0}(0,0,0) + O(L^{-3}) \\ &= (\alpha+s)(\beta+u) + O(L^{-3})\end{aligned}$$

since $A_{0,0}(z,z,z) = 1$ for all z, in particular for $z = 0$. By the use of Lemma 6.2.2, we can deform the line integrals into contour integrals around circles of radius 1 around the origin. Thus,

$$J_{11} = \sum_{n\leqslant y_1}\left(\frac{1}{2\pi i}\right)^2 \oint\left(\frac{y_1}{n}\right)^s\frac{(s+\alpha)ds}{s^{i+1}}\oint\left(\frac{y_1}{n}\right)^s\frac{(u+\beta)du}{u^{j+1}} + O(L^{i+j-2}).$$

These integrals can be computed by the use of (6.13), so that

$$J_{11} = \frac{1}{i!j!}\frac{d^2}{dxdy}e^{\alpha x+\beta y}\sum_{n\leqslant y_1}\frac{1}{n}\left(x+\log\frac{y_1}{n}\right)^i\left(y+\log\frac{y_1}{n}\right)^j\bigg|_{x=y=0} + O(L^{i+j-2}).$$

Let us note that

$$\frac{d}{dx}e^{\alpha x}\sum_{n\leqslant y_1}\frac{1}{n}\left(x+\log\frac{y_1}{n}\right)^i\bigg|_{x=0} = \frac{\log^i y_1}{\log y_1}\frac{d}{dx}y_1^{\alpha x}\left(x+\frac{\log(y_1/n)}{\log y_1}\right)^i\bigg|_{x=0}.$$

Now sum over i to obtain

$$P_1[n] = \sum_i a_i\left(x+\frac{\log(y_1/n)}{\log y_1}\right)^i$$

and similarly over j so that

$$\begin{aligned}I_{11}'(\alpha,\beta) &= \widehat{w}(0)\zeta(1+\alpha+\beta)\sum_{i,j}\frac{a_i a_j}{\log^2 y_1} \\ &\quad\times\frac{d^2}{dxdy}\left[y_1^{\alpha x+\beta y}\sum_{n\leqslant y_1}\frac{1}{n}\left(x+\frac{\log(y_1/n)}{\log y_1}\right)^i\left(y+\frac{\log(y_1/n)}{\log y_1}\right)^j\bigg|_{x=y=0}\right] \\ &\quad + O(T/L) \\ &= \frac{\widehat{w}(0)}{(\alpha+\beta)\log^2 y_1}\frac{d^2}{dxdy}\Bigg[y_1^{\alpha x+\beta y} \\ &\quad\times\sum_{n\leqslant y_1}\frac{1}{n}P\left(x+\frac{\log(y_1/n)}{\log y_1}\right)P\left(y+\frac{\log(y_1/n)}{\log y_1}\right)\bigg|_{x=y=0}\Bigg] + O(T/L)\end{aligned}$$

$$
\begin{aligned}
&= \frac{\widehat{w}(0)}{(\alpha+\beta)\log^2 y_1} \frac{d^2}{dxdy}\Bigg[y_1^{\alpha x+\beta y} \\
&\quad\times \int_1^{y_1} r^{-1} P\left(x+\frac{\log(y_1/r)}{\log y_1}\right) P\left(y+\frac{\log(y_1/r)}{\log y_1}\right) dr\bigg|_{x=y=0}\Bigg] + O(T/L) \\
&= \frac{\widehat{w}(0)}{(\alpha+\beta)\log y_1} \frac{d^2}{dxdy}\left[y_1^{\alpha x+\beta y} \int_0^1 P(x+u)P(y+u)du\bigg|_{x=y=0}\right] + O(T/L).
\end{aligned}
$$

In the second equality we made use of $\zeta(1+\alpha+\beta) = 1/(\alpha+\beta)+O(1)$, in the third equality we used the Euler-MacLaurin formula, and the in the fourth equality we employed the change of variables $r = M^{1-u}$. By adding and subtracting the same quantity we find that

$$
I_{11}(\alpha,\beta) = [I'_{11}(\alpha,\beta)+I'_{11}(-\beta,-\alpha)] + I'_{11}(-\beta,-\alpha)(T^{-\alpha-\beta}-1) + O(T/L). \tag{6.17}
$$

For the term in square brackets we have

$$
\begin{aligned}
&c'_{11}(\alpha,\beta) + c'_{11}(-\beta,-\alpha) \\
&= \frac{1}{(\alpha+\beta)\log y_1} \int_0^1 (P'(u)+\alpha P(u)\log y_1)(P'(u)+\beta P(u)\log y_1)du \\
&\quad - \frac{1}{(\alpha+\beta)\log y_1} \int_0^1 (P'(u)-\beta P(u)\log y_1)(P'(u)-\alpha P(u)\log y_1)du \\
&= \int_0^1 2P'(u)P(u)du = 1.
\end{aligned}
$$

For the other term in (6.17) we have

$$
\begin{aligned}
&c'_{11}(-\beta,-\alpha)(T^{-\alpha-\beta}-1) \\
&= \frac{T^{-\alpha-\beta}-1}{(-\beta-\alpha)\log y_1} \frac{d^2}{dxdy} y_1^{-\beta x-\alpha y} \int_0^1 P(x+u)P(y+u)du\bigg|_{x=y=0} \\
&= \frac{1-T^{-\alpha-\beta}}{(\alpha+\beta)\log y_1} \frac{d^2}{dxdy} y_1^{-\beta x-\alpha y} \int_0^1 P(x+u)P(y+u)du\bigg|_{x=y=0} \\
&= \frac{1}{\theta_1} \frac{d^2}{dxdy} y_1^{-\beta x-\alpha y} \int_0^1 \int_0^1 T^{-v(\alpha+\beta)} P(x+u)P(y+u)dudv\bigg|_{x=y=0},
\end{aligned}
$$

by the use of

$$
\frac{1-T^{-\alpha-\beta}}{(\alpha+\beta)\log y_1} = \frac{1}{\theta_1}\int_0^1 T^{-v(\alpha+\beta)}dv. \tag{6.18}
$$

The additional restriction that $|\alpha+\beta| \gg L^{-1}$ is dealt with the holomorphy of $I(\alpha,\beta)$ and $c(\alpha,\beta)$ with $\alpha,\beta \ll L^{-1}$ which implies that the error term is also holomorphic in this region. The maximum modulus principle extends the error term to this enlarged domain. This proves Lemma 6.1.1.

6.3.2 The mean value integral $I_{12}(\alpha,\beta)$

Let us follow the same strategy as in $I_{11}(\alpha,\beta)$. We first insert the definitions of ψ_1 and ψ_2 into the mean value integral I_{12} so that

$$\begin{aligned}
I_{12}(\alpha,\beta) &= \int_{-\infty}^{\infty} w(t)\zeta(\tfrac{1}{2}+\alpha+it)\zeta(\tfrac{1}{2}+\beta-it)\overline{\psi_1}\psi_2(\sigma_0+it)dt \\
&= \int_{-\infty}^{\infty} w(t)\zeta(\tfrac{1}{2}+\alpha+it)\zeta(\tfrac{1}{2}+\beta-it) \\
&\quad\times \sum_{h\le y_1}\frac{\mu(h)}{h^{1/2-it}}P_1[h]\sum_{k\le y_2}\frac{\mu(k)}{k^{1/2+it}}\sum_{\ell=2}^{K}\sum_{p_1\cdots p_\ell|k}\frac{\log p_1\cdots\log p_\ell}{\log^\ell y_2}P_\ell[k]dt \\
&= \sum_{\ell=2}^{K}\sum_{h,k}\frac{\mu(h)\mu(k)}{(hk)^{1/2}}P_1[h]\sum_{p_1\cdots p_\ell|k}\frac{\log p_1\cdots\log p_\ell}{\log^\ell y_2}P_\ell[k] \\
&\quad\times \int_{-\infty}^{\infty} w(t)\zeta(\tfrac{1}{2}+\alpha+it)\zeta(\tfrac{1}{2}+\beta-it)\left(\frac{k}{h}\right)^{-it}dt.
\end{aligned}$$

As for $I_{11}(\alpha,\beta)$, we use at this point Lemma 6.2.1 to write $I_{12}(\alpha,\beta) = I'_{12}(\alpha,\beta) + I''_{12}(\alpha,\beta) + E(\alpha,\beta)$ with

$$\begin{aligned}
I'_{12}(\alpha,\beta) &= \sum_{\ell=2}^{K}\sum_{h,k}\frac{\mu(h)\mu(k)}{(hk)^{1/2}}P_1[h]\sum_{p_1\cdots p_\ell|k}\frac{\log p_1\cdots\log p_\ell}{\log^\ell y_2}P_\ell[k] \\
&\quad\times \sum_{hm=kn}\frac{1}{m^{1/2+\alpha}n^{1/2+\beta}}\int_{-\infty}^{\infty}V_{\alpha,\beta}(mn,t)w(t)dt, \qquad (6.19)
\end{aligned}$$

and

$$\begin{aligned}
E(\alpha,\beta) &\ll_{A,\theta_1,\theta_2} T^{-A}\sum_{\ell=2}^{K}\sum_{h,k}\frac{\mu(h)\mu(k)}{(hk)^{1/2}}P_1[h]\sum_{p_1\cdots p_\ell|k}\frac{\log p_1\cdots\log p_\ell}{\log^\ell y_2}P_\ell[k] \\
&\ll T^{-A}\sum_{\ell=2}^{K}\sum_{h\le y_1}\sum_{k\le y_2}\frac{1}{(hk)^{1/2}}\sum_{p_1\cdots p_\ell|k}1 \ll T^{-A}\sum_{\ell=2}^{K}\sum_{h\le y_1}\sum_{k\le y_2}\frac{(d(k))^\ell}{(hk)^{1/2}} \\
&\ll T^{-A}\sum_{h\le y_1}\frac{1}{h^{1/2-\varepsilon}}\sum_{k\le y_2}\frac{1}{k^{1/2-\varepsilon}} \ll T^{-A}y_1^{1/2-\varepsilon}y_2^{1/2-\varepsilon} \\
&= T^{-A}T^{\theta_1(1/2-\varepsilon)\theta_2(1/2-\varepsilon)} = T^{-A+(\theta_1+\theta_2)/2-\varepsilon}
\end{aligned}$$

for any $A>2$. We remark that the above computation works for $\theta_1+\theta_2$ arbitrarily large but the error term T^{-A} coming from Lemma 6.2.1 is only valid for $\theta_1+\theta_2<1$. For reasons of symmetry, $I''_{12}(\alpha,\beta)$ can be obtained from $I'_{12}(\alpha,\beta)$ by switching α and $-\beta$ and multiplying by

$$\left(\frac{t}{2\pi}\right)^{-\alpha-\beta} = T^{-\alpha-\beta} + O(L^{-1}),$$

for $t \asymp T$. This means that we may concentrate our efforts on $I_{12}'(\alpha,\beta)$. The next step is to use the Mellin integral representations of the polynomials P_1

$$P_1[h] = \sum_i \frac{a_i}{\log^i y_1}(\log(y_1/h))^i = \sum_i \frac{a_i i!}{\log^i y_1}\frac{1}{2\pi i}\int_{(1)} \left(\frac{y_1}{h}\right)^s \frac{ds}{s^{i+1}},$$

and P_ℓ

$$P_\ell[k] = \sum_j \frac{b_{\ell,j}}{\log^j y_2}(\log(y_2/k))^j = \sum_j \frac{b_{\ell,j} j!}{\log^j y_2}\frac{1}{2\pi i}\int_{(1)} \left(\frac{y_2}{k}\right)^u \frac{du}{u^{j+1}},$$

and the definition of $V_{\alpha,\beta}$ in Lemma 6.2.1 to write

$$\begin{aligned} I_{12}'(\alpha,\beta) &= \int_{-\infty}^{\infty} w(t) \sum_{\ell=2}^{L}\sum_{i,j} \frac{a_i b_{\ell,j} i! j!}{\log^i y_1 \log^{j+\ell} y_2} \\ &\times \sum_{km=hn} \frac{\mu(h)\mu(k)}{(hk)^{1/2} m^{1/2+\alpha} n^{1/2+\beta}} \sum_{p_1\cdots p_\ell | k} \log p_1 \cdots \log p_\ell \\ &\times \left(\frac{1}{2\pi i}\right)^3 \int_{(1)}\int_{(1)}\int_{(1)} \left(\frac{y_1}{h}\right)^s \left(\frac{y_2}{k}\right)^u \frac{g_{\alpha,\beta}(z,t)}{(mn)^z}\frac{G(z)}{z}dz \frac{ds}{s^{i+1}}\frac{du}{u^{j+1}} dt. \end{aligned}$$

We now have to compute the arithmetical sum $\sum_{km=hn}$. Further details on this procedure can be found in [RRZ16]. Let us define

$$S_\ell = S_{\ell,\alpha,\beta}(s,u,z) = \sum_{km=hn} \frac{\mu(h)\mu(k)}{(hk)^{1/2} m^{1/2+\alpha} n^{1/2+\beta}} \sum_{p_1\cdots p_\ell | k} \log p_1 \cdots \log p_\ell.$$

We start by inverting the order of the sum so that

$$\begin{aligned} S_\ell &= (-1)^\ell \sum_{\substack{p_i \neq p_j \\ i<j}} \log p_1 \cdots \log p_\ell \sum_{\substack{hn = p_1\cdots p_\ell \tilde{k} m \\ (p_1\cdots p_\ell, \tilde{k})=1}} \frac{\mu(h)\mu(\tilde{k})}{h^{1/2+s}\tilde{k}^{1/2+u} m^{1/2+\alpha+z} n^{1/2+\beta+z}} \frac{1}{(p_1\cdots p_\ell)^{1/2+u}} \\ &= (-1)^\ell \sum_{\substack{p_i \neq p_j \\ i<j}} \frac{\log p_1 \cdots \log p_\ell}{(p_1\cdots p_\ell)^{1/2+u}} \tilde{S}_{\ell,\alpha,\beta}(s,u,z), \end{aligned} \tag{6.20}$$

where $k = \tilde{k} p_1 \cdots p_\ell$ and where we define the inner sum to be

$$\tilde{S}_\ell = \tilde{S}_{\ell,\alpha,\beta}(s,u,z) = \sum_{\substack{h,\tilde{k},m,n \\ hn = p_1\cdots p_\ell \tilde{k} m \\ (p_1\cdots p_\ell, \tilde{k})=1}} \frac{\mu(h)\mu(\tilde{k})}{h^{1/2+s}\tilde{k}^{1/2+u} m^{1/2+\alpha+z} n^{1/2+\beta+z}}.$$

Recall that $\nu_p(n) = n'$ denotes the number of times the prime number p appears in n. We can write the above as

$$\begin{aligned} \tilde{S}_\ell = &\prod_{p\in\{p_1,\cdots,p_\ell\}} \sum_{h'+n'=m'+1} \frac{\mu(p^{h'})}{(p^{h'})^{1/2+s}(p^{m'})^{1/2+\alpha+z}(p^{n'})^{1/2+\beta+z}} \\ &\times \prod_{p\notin\{p_1,\cdots,p_\ell\}} \sum_{h'+n'=k'+m'} \frac{\mu(p^{h'})\mu(p^{k'})}{(p^{h'})^{1/2+s}(p^{k'})^{1/2+u}(p^{m'})^{1/2+\alpha+z}(p^{n'})^{1/2+\beta+z}} \end{aligned}$$

$$= \frac{\Pi_1(\alpha,\beta,s,u,z)}{\Pi_2(\alpha,\beta,s,u,z)} \Pi_3(\alpha,\beta,s,u,z), \tag{6.21}$$

where we define

$$\Pi_1(\alpha,\beta,s,u,z) = \prod_p \sum_{h'+n'=k'+m'} \frac{\mu(p^{h'})\mu(p^{k'})}{(p^{h'})^{1/2+s}(p^{k'})^{1/2+u}(p^{m'})^{1/2+\alpha+z}(p^{n'})^{1/2+\beta+z}}$$
$$= \prod_p \left(1 + \frac{1}{p^{1+s+u}} - \frac{1}{p^{1+s+\alpha+z}} - \frac{1}{p^{1+u+\beta+z}} + \frac{1}{p^{1+\alpha+\beta+2z}} + O(p^{-2+\varepsilon})\right),$$

as well as

$$\Pi_2(\alpha,\beta,s,u,z)$$
$$= \prod_{p\in\{p_1,\cdots,p_\ell\}} \sum_{h'+n'=k'+m'} \frac{\mu(p^{h'})\mu(p^{k'})}{(p^{h'})^{1/2+s}(p^{k'})^{1/2+u}(p^{m'})^{1/2+\alpha+z}(p^{n'})^{1/2+\beta+z}}$$
$$= \prod_{p\in\{p_1,\cdots,p_\ell\}} \left(1 + \frac{1}{p^{1+s+u}} - \frac{1}{p^{1+s+\alpha+z}} - \frac{1}{p^{1+u+\beta+z}} + \frac{1}{p^{1+\alpha+\beta+2z}} + O(p^{-2+\varepsilon})\right),$$

and finally

$$\Pi_3(\alpha,\beta,s,u,z) = \prod_{p\in\{p_1,\cdots,p_\ell\}} \sum_{h'+n'=m'+1} \frac{\mu(p^{h'})}{(p^{h'})^{1/2+s}(p^{m'})^{1/2+\alpha+z}(p^{n'})^{1/2+\beta+z}}$$
$$= \prod_{p\in\{p_1,\cdots,p_\ell\}} \left(\frac{1}{p^{1/2+\beta+z}} - \frac{1}{p^{1/2+s}} + O(p^{-2+\varepsilon})\right).$$

Hence we arrive at the following expression for $\tilde{S}_\ell$

$$\tilde{S}_\ell = \prod_p \left(1 + \frac{1}{p^{1+s+u}} - \frac{1}{p^{1+s+\alpha+z}} - \frac{1}{p^{1+u+\beta+z}} + \frac{1}{p^{1+\alpha+\beta+2z}} + O(p^{-2+\varepsilon})\right)$$
$$= \frac{\zeta(1+s+u)\zeta(1+\alpha+\beta+2z)}{\zeta(1+u+\beta+z)\zeta(1+s+\alpha+z)} A_{\alpha,\beta}(s,u,z),$$

where the arithmetical factor $A_{\alpha,\beta}(s,u,z)$ is given by an absolutely convergent Euler product in some product of half-planes containing the origin. Therefore, when we return to the expression for S_ℓ in (6.20), we obtain the following

$$S_\ell = \frac{\zeta(1+s+u)\zeta(1+\alpha+\beta+2z)}{\zeta(1+u+\beta+z)\zeta(1+s+\alpha+z)} A_{\alpha,\beta}(s,u,z)(-1)^\ell \sum_{\substack{p_i\neq p_j\\ i<j}} \log p_1 \cdots \log p_\ell$$
$$\times \prod_{p\in\{p_1,\cdots,p_\ell\}} \frac{E(p)+O(p^{-2+\varepsilon})}{1-\frac{1}{p^{1+s+\alpha+z}}+\frac{1}{p^{1+\alpha+\beta+2z}} - E(p) + O(p^{-2+\varepsilon})}, \tag{6.22}$$

where
$$E(p) = \frac{1}{p^{1/2+u}}\left(-\frac{1}{p^{1/2+s}} + \frac{1}{p^{1/2+\beta+z}}\right) = -\frac{1}{p^{1+s+u}} + \frac{1}{p^{1+\beta+u+z}}.$$

At this stage, we compare (6.22) in its exact form (that is, with big-O terms replaced by their exact expressions) against (6.20) and (6.21) in its exact form, and we use the fact that for $\alpha = \beta = 0$ and $s = u = z$, the ratio of zeta functions

$$\frac{\zeta(1+s+u)\zeta(1+\alpha+\beta+2z)}{\zeta(1+u+\beta+z)\zeta(1+s+\alpha+z)}$$

reduces to 1. In other words, reverting the p-adic analysis in

$$\begin{aligned}&\frac{\zeta(1+s+u)\zeta(1+\alpha+\beta+2z)}{\zeta(1+u+\beta+z)\zeta(1+s+\alpha+z)}A_{\alpha,\beta}(s,u,z)\\ &=\prod_p \sum_{h'+n'=k'+m'} \frac{\mu(p^{h'})\mu(p^{k'})}{(p^{h'})^{1/2+s}(p^{k'})^{1/2+u}(p^{m'})^{1/2+\alpha+z}(p^{n'})^{1/2+\beta+z}},\end{aligned}$$

we find that

$$\frac{\zeta(1+s+u)\zeta(1+\alpha+\beta+2z)}{\zeta(1+u+\beta+z)\zeta(1+s+\alpha+z)}A_{\alpha,\beta}(s,u,z) = \sum_{hn=km} \frac{\mu(h)\mu(k)}{h^{1/2+s}k^{1/2+u}m^{1/2+\alpha+z}n^{1/2+\beta+z}}.$$

Following (6.16), we know that

$$A_{0,0}(z,z,z) = \sum_{km=hn} \frac{\mu(h)\mu(k)}{(hkmn)^{1/2+z}},$$

and thus, we find that

$$A_{0,0}(z,z,z) = 1$$

for all z. Let us denote the last part of (6.22) by H_ℓ; specifically

$$\begin{aligned}H_\ell &= (-1)^\ell \sum_{\substack{p_i\neq p_j\\ i<j}} \prod_{p\in\{p_1,\cdots,p_\ell\}} (E(p)+O(p^{-2+\varepsilon}))\log p\\ &\quad\times\left(1+E(p)+\frac{1}{p^{1+s+\alpha+z}}-\frac{1}{p^{1+\alpha+\beta+2z}}+O(p^{-2+\varepsilon})\right)\\ &= (-1)^\ell \sum_{\substack{p_i\neq p_j\\ i<j}} \prod_{p\in\{p_1,\cdots,p_\ell\}} \left(E(p)\log p + O\left(\frac{\log p}{p^{2-\varepsilon}}\right)\right).\end{aligned}$$

We now employ the principle of inclusion-exclusion to write

$$H_\ell = (-1)^\ell\left(\sum_{p\in\{p_1,\cdots,p_\ell\}} E(p)\log p + O\left(\frac{\log p}{p^{2-\varepsilon}}\right)\right)^\ell + \sum_{p\in\{p_1,\cdots,p_\ell\}} B(p),$$

where

$$B(p) \ll_{\alpha,\beta,s,u,z,\varepsilon} \frac{1}{p^{2-\varepsilon}}.$$

To complete the computation, we must identify the logarithms of the prime numbers with the signature of the von Mangoldt function $\Lambda(n)$ and hence match the resulting expressions to logarithmic derivatives of the Riemann zeta-function by the use of

$$\frac{\zeta'}{\zeta}(s) = -\sum_{n=1}^{\infty}\Lambda(n)n^{-s} = -\sum_p \frac{\log p}{p^s}\left(1-\frac{1}{p^s}\right)^{-1} = -\sum_p \frac{\log p}{p^s} + O\left(\frac{\log p}{p^{2s}}\right),$$

for $\mathrm{Re}(s) > 1$. With this in mind, H_ℓ becomes

$$\begin{aligned} H_\ell &= (-1)^\ell \left(\frac{\zeta'}{\zeta}(1+s+u) - \frac{\zeta'}{\zeta}(1+\beta+u+z) + O(1) \right)^\ell + D(\alpha,\beta,s,u,z) \\ &= (-U)^\ell + \sum_{m=0}^{\ell-1} U^m B_m(\alpha,\beta,s,u,z) + D(\alpha,\beta,s,u,z), \end{aligned}$$

where $D(\alpha,\beta,s,u,z)$ contains terms of smaller order and where

$$U = -\frac{\zeta'}{\zeta}(1+s+u) + \frac{\zeta'}{\zeta}(1+\beta+u+z).$$

We also have that

$$B_m(\alpha,\beta,s,u,z) \ll_{\alpha,\beta,s,u,z} \sum_p \frac{\log p}{p^{2-\varepsilon}}.$$

All of these terms are analytic in a larger region of the complex plane, thus we are only interested in the term U^ℓ. Consequently, the end result of this is that

$$\begin{aligned} I'_{12}(\alpha,\beta) = \int_{-\infty}^{\infty} w(t) \sum_{\ell=2}^{K} \sum_{i,j} & \frac{a_i b_{\ell,j} i! j!}{\log^i y_1 \log^{j+\ell} y_2} \left(\frac{1}{2\pi i}\right)^3 \int_{(1)} \int_{(1)} \int_{(1)} \\ &\times \frac{\zeta(1+s+u)\zeta(1+\alpha+\beta+2z)}{\zeta(1+u+\beta+z)\zeta(1+s+\alpha+z)} A_{\alpha,\beta}(s,u,z) \\ &\times \left(\frac{\zeta'}{\zeta}(1+s+u) - \frac{\zeta'}{\zeta}(1+\beta+u+z) \right)^\ell \\ &\times (-1)^\ell y_1^s y_2^u \frac{G(z)}{z} g_{\alpha,\beta}(z,t) \frac{ds}{s^{i+1}} \frac{du}{u^{j+1}} dt. \end{aligned}$$

The next step is to deform the path of integration to $\mathrm{Re}(z) = -\delta + \varepsilon$ where $\delta > 0$ is small and $\mathrm{Re}(s) = \mathrm{Re}(u) = \delta$. By doing this, we pick up the contribution of the residue of the simple pole of $1/z$ at $z = 0$ only, since, as before in the $I_{11}(\alpha,\beta)$ case, $G(z)$ vanishes at the pole of $\zeta(1+\alpha+\beta+2z)$. The new path of integration contributes

$$\ll T^{1+\varepsilon} \left(\frac{y_1 y_2}{T} \right)^\delta \ll T^{1-\varepsilon}. \tag{6.23}$$

by keeping $\theta_1 + \theta_2 = 1 - \varepsilon$ (since $y_1 = T^{\theta_1}$ and $y_2 = T^{\theta_2}$). We now write

$$I'_{12}(\alpha,\beta) = I'_{120}(\alpha,\beta) + O(T^{-1+\varepsilon}),$$

where $I'_{120}(\alpha,\beta)$ corresponds to the residue at $z = 0$. Then

$$\begin{aligned} I'_{120}(\alpha,\beta) = \int_{-\infty}^{\infty} w(t) \sum_{\ell=2}^{K} \sum_{i,j} & \frac{a_i b_{\ell,j} i! j!}{\log^i y_1 \log^{j+\ell} y_2} \\ &\times \left(\frac{1}{2\pi i}\right)^2 \int_{(\delta)} \int_{(\delta)} \operatorname*{Res}_{z=0} \frac{G(z)}{z} g_{\alpha,\beta}(z,t) y_1^s y_2^u \\ &\times \frac{\zeta(1+s+u)\zeta(1+\alpha+\beta+2z)}{\zeta(1+s+\alpha+z)\zeta(1+u+\beta+z)} A_{\alpha,\beta}(s,u,z) \end{aligned}$$

$$\begin{aligned}&\times(-1)^\ell\bigg(\frac{\zeta'}{\zeta}(1+s+u)-\frac{\zeta'}{\zeta}(1+\beta+u+z)\bigg)^\ell\frac{ds}{s^{i+1}}\frac{du}{u^{j+1}}dt\\ &=\widehat{w}(0)\zeta(1+\alpha+\beta)\sum_{\ell=2}^{K}(-1)^\ell\sum_{i,j}\frac{a_ib_{\ell,j}i!j!}{\log^i y_1\log^{j+\ell}y_2}J_{12},\end{aligned}\tag{6.24}$$

where

$$\begin{aligned}J_{12}(\alpha,\beta)=\bigg(\frac{1}{2\pi i}\bigg)^2\int_{(\delta)}\int_{(\delta)}\frac{\zeta(1+s+u)A_{\alpha,\beta}(s,u,0)}{\zeta(1+u+\beta)\zeta(1+s+\alpha)}\bigg(\frac{\zeta'}{\zeta}(1+s+u)-\frac{\zeta'}{\zeta}(1+\beta+u)\bigg)^\ell\\ \times y_1^sy_2^u\frac{ds}{s^{i+1}}\frac{du}{u^{j+1}}.\end{aligned}$$

Let us now use the binomial theorem to write

$$\begin{aligned}&J_{12}(\alpha,\beta)\\ &=\bigg(\frac{1}{2\pi i}\bigg)^2\int_{(\delta)}\int_{(\delta)}\frac{\zeta(1+s+u)A_{\alpha,\beta}(s,u,0)}{\zeta(1+\beta+u)\zeta(1+\alpha+s)}\\ &\times\sum_{r=0}^{\ell}\binom{\ell}{r}\bigg(\frac{\zeta'}{\zeta}(1+\beta+u)\bigg)^{\ell-r}\bigg(-\frac{\zeta'}{\zeta}(1+s+u)\bigg)^r y_1^sy_2^u\frac{ds}{s^{i+1}}\frac{du}{u^{j+1}}\\ &=\bigg(\frac{1}{2\pi i}\bigg)^2\int_{(\delta)}\int_{(\delta)}\frac{A_{\alpha,\beta}(s,u,0)}{\zeta(1+\beta+u)\zeta(1+\alpha+s)}\sum_{r=0}^{\ell}\binom{\ell}{r}\bigg(\frac{\zeta'}{\zeta}(1+\beta+u)\bigg)^{\ell-r}\\ &\times\sum_{n=1}^{\infty}\frac{(\mathbf{1}*\Lambda^{*r})(n)}{n^{1+s+u}}y_1^sy_2^u\frac{ds}{s^{i+1}}\frac{du}{u^{j+1}}\\ &=\sum_{n\leqslant\min(y_1,y_2)}\sum_{r=0}^{\ell}\binom{\ell}{r}\frac{(\mathbf{1}*\Lambda^{*r})(n)}{n}\bigg(\frac{1}{2\pi i}\bigg)^2\\ &\times\int_{(\delta)}\int_{(\delta)}\frac{A_{\alpha,\beta}(s,u,0)}{\zeta(1+\beta+u)\zeta(1+\alpha+s)}\bigg(\frac{\zeta'}{\zeta}(1+\beta+u)\bigg)^{\ell-r}\bigg(\frac{y_1}{n}\bigg)^s\bigg(\frac{y_2}{n}\bigg)^u\frac{ds}{s^{i+1}}\frac{du}{u^{j+1}},\end{aligned}$$

where we have used the Dirichlet convolution of

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s}\quad\text{and}\quad-\frac{\zeta'}{\zeta}(s)=\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s},$$

for $\mathrm{Re}(s)>1$. Here $\mathbf{1}(n)=1$ for all n denotes the identity function. Next, we take $\delta\asymp L^{-1}$ and bound the integral trivially to get $J_{12}\ll L^{i+j-1}$. This means that we can use a Taylor series so that $A_{\alpha,\beta}(s,u,0)=A_{0,0}(0,0,0)+O(|s|+|u|)$ to write $J_{12}(\alpha,\beta)=J_{12}'(\alpha,\beta)+O(L^{i+j-2})$, say. We recall that we have shown earlier that $A_{0,0}(z,z,z)=1$ for all z, in particular $A_{0,0}(0,0,0)=1$. This implies that the complex variables s and u are now separated as

$$J_{12}'(\alpha,\beta)=\sum_{n\leqslant\min(y_1,y_2)}\sum_{r=0}^{\ell}\binom{\ell}{r}\frac{(\mathbf{1}*\Lambda^{*r})(n)}{n}L_{12,1}L_{12,2},$$

where

$$L_{12,1}=\frac{1}{2\pi i}\int_{(\delta)}\frac{1}{\zeta(1+\alpha+s)}\bigg(\frac{y_1}{n}\bigg)^s\frac{ds}{s^{i+1}},$$

and

$$L_{12,2} = \frac{1}{2\pi i}\int_{(\delta)} \frac{1}{\zeta(1+\beta+u)}\left(\frac{\zeta'}{\zeta}(1+\beta+u)\right)^{\ell-r}\left(\frac{y_2}{n}\right)^u \frac{du}{u^{j+1}}. \tag{6.25}$$

The first of these two integrals is dealt with in [You10] and its main term is

$$L_{12,1} = \frac{1}{2\pi i}\oint (\alpha+s)\left(\frac{y_1}{n}\right)^s \frac{ds}{s^{i+1}} = \frac{1}{i!}\frac{d}{dx}e^{\alpha x}\left(x+\log\frac{y_1}{n}\right)^i\bigg|_{x=0}.$$

For the second integral we will need the following Lemma 6.2.2 and equation (6.13). Hence, one gets

$$L_{12,2} = \frac{(-1)^{\ell-r}}{2\pi i}\oint (\beta+u)^{1-\ell+r}\left(\frac{y_2}{n}\right)^u \frac{du}{u^{j+1}} = \frac{(-1)^{\ell-r}}{j!}\frac{d^{1-\ell+r}}{dy^{1-\ell+r}}e^{\beta y}\left(y+\log\frac{y_2}{n}\right)^j\bigg|_{y=0}.$$

This means that when we insert these results into J'_{12} we obtain

$$\begin{aligned} J'_{12}(\alpha,\beta) &= \frac{1}{i!}\frac{1}{j!}\sum_{n\leqslant \min(y_1,y_2)}\sum_{r=0}^{\ell}(-1)^{\ell-r}\binom{\ell}{r}\frac{(\mathbf{1}*\Lambda^{*r})(n)}{n} \\ &\times \frac{d}{dx}\frac{d^{1-\ell+r}}{dy^{1-\ell+r}}e^{\alpha x+\beta y}\left(x+\log\frac{y_1}{n}\right)^i\bigg|_{x=0}\left(y+\log\frac{y_2}{n}\right)^j\bigg|_{y=0} + O(L^{i+j-2}). \end{aligned}$$

By making the changes

$$x \to \frac{x}{\log y_1} \quad \text{and} \quad y \to \frac{y}{\log y_2},$$

we can write this in the more convenient form

$$\begin{aligned} J'_{12}(\alpha,\beta) &= \frac{\log^{i-1}y_1\log^{j-1}y_2}{i!j!}\sum_{n\leqslant \min(y_1,y_2)}\sum_{r=0}^{\ell}(-1)^{\ell-r}\binom{\ell}{r}\frac{(\mathbf{1}*\Lambda^{*r})(n)}{n} \\ &\times \frac{d}{dx}\frac{d^{1-\ell+r}}{dy^{1-\ell+r}}y_1^{\alpha x}y_2^{\beta y}\left(x+\frac{\log(y_1/n)}{\log y_1}\right)^i\bigg|_{x=0}\left(y+\frac{\log(y_2/n)}{\log y_2}\right)^j\bigg|_{y=0} \\ &+ O(L^{i+j-2}). \end{aligned}$$

Telescoping back to (6.24) we find that

$$\begin{aligned} I'_{120}(\alpha,\beta) &= \frac{\widehat{w}(0)}{(\alpha+\beta)\log y_1\log y_2}\frac{d^2}{dxdy}\Bigg[y_1^{\alpha x}y_2^{\beta y} \\ &\times \sum_{\ell=2}^{L}(-1)^{\ell}\frac{1}{\log^{\ell}y_2}\sum_{r=0}^{\ell}(-1)^{\ell-r}\binom{\ell}{r}\frac{d^{r-\ell}}{dy^{r-\ell}} \\ &\times \sum_{n\leqslant \min(y_1,y_2)}\frac{(\mathbf{1}*\Lambda^{*r})(n)}{n}P_1\left(x+\frac{\log(y_1/n)}{\log y_1}\right)P_\ell\left(y+\frac{\log(y_2/n)}{\log y_2}\right)\bigg|_{x=y=0}\Bigg] \\ &+ O(T/L), \end{aligned}$$

where the sum over i has been identified to the polynomial P_1, and the sum over j to the polynomials P_ℓ. We now perform the summation over n by using Lemma 6.2.5. To do so, we now set $y_1 \geq y_2$. The lemma yields

$$\begin{aligned}\sum_{n\leqslant y_2} \frac{(\mathbf{1} * \Lambda^{*r})(n)}{n^{1+s}} P_1\left(x+\frac{\log(y_1/n)}{\log y_1}\right) P_\ell\left(y+\frac{\log(y_2/n)}{\log y_2}\right) \\ = \frac{\log^{r+1} y_2}{y_2^s}\int_0^1 (1-u)^r P_1\left(x+1-(1-u)\frac{\theta_2}{\theta_1}\right) P_\ell(y+u) y_2^{us} du \\ + O(\log(3y_2)^r).\end{aligned}$$

Therefore, the resulting expression for I'_{120} is

$$\begin{aligned}I'_{120}(\alpha,\beta) &= \frac{\widehat{w}(0)}{(\alpha+\beta)\log y_1 \log y_2}\frac{d^2}{dxdy}\Bigg[\int_0^1 \\ &\times y_1^{\alpha x} y_2^{\beta y}\sum_{\ell=2}^{L}(-1)^\ell \frac{1}{\log^\ell y_2}\sum_{r=0}^{\ell}(-1)^{\ell-r}\binom{\ell}{r}\frac{d^{r-\ell}}{dy^{r-\ell}} \\ &\times \frac{\log^{r+1} y_2}{r!}(1-u)^r P_1\left(x+1-(1-u)\frac{\theta_2}{\theta_1}\right) P_\ell(y+u)du\Bigg|_{x=y=0}\Bigg] + O(T/L).\end{aligned}$$

Now we must return to I_{12}. We recall that $I_{12}(\alpha,\beta)$ was formed by adding $I'_{12}(\alpha,\beta)$ and $I''_{12}(\alpha,\beta)$, where I''_{12} is formed by taking I'_{12}, switching α and $-\beta$, and then multiplying by $T^{-\alpha-\beta}$. Note that $r \leq \ell$ and thus only the case $r=\ell$ contributes to the main term. Therefore

$$\begin{aligned}I'_{12}(\alpha,\beta) &= \frac{\widehat{w}(0)}{(\alpha+\beta)\log y_1}\sum_{\ell=2}^{L}(-1)^\ell\frac{1}{\ell!}\frac{d^2}{dxdy}\Bigg[y_1^{\alpha x}y_2^{\beta y} \\ &\times \int_0^1 (1-u)^\ell P_1\left(x+1-(1-u)\frac{\theta_2}{\theta_1}\right)P_\ell(y+u)du\Bigg|_{x=y=0}\Bigg] + O(T/L).\end{aligned}$$

We now use

$$\begin{aligned}I_{12}(\alpha,\beta) &= I'_{12}(\alpha,\beta) + T^{-\alpha-\beta}I'_{12}(-\beta,-\alpha) + O(T/L) \\ &= \big(I'_{12}(\alpha,\beta)+I'_{12}(-\beta,-\alpha)\big) + \big(T^{-\alpha-\beta}-1\big)I'_{12}(-\beta,-\alpha) + O(T/L).\end{aligned}$$

We first take a look at the first term in the brackets

$$\begin{aligned}&\frac{d^2}{dxdy}\Bigg[\left(y_1^{\alpha x}y_2^{\beta y} - y_1^{-\beta x}y_2^{-\alpha y}\right)\int_0^1 (1-u)^\ell P_1\left(x+1-(1-u)\frac{\theta_2}{\theta_1}\right)P_\ell(y+u)du\Bigg|_{x=y=0}\Bigg] \\ &= (\alpha+\beta)\log y_1 \int_0^1 (1-u)^\ell P_1\left(1-(1-u)\frac{\theta_2}{\theta_1}\right)P'_\ell(y)du \\ &\quad + (\alpha+\beta)\log y_2\int_0^1 (1-u)^\ell P'_1\left(1-(1-u)\frac{\theta_2}{\theta_1}\right)P_\ell(y)du \\ &= (\alpha+\beta)\log y_1\Bigg(\int_0^1 (1-u)^\ell P_1\left(1-(1-u)\frac{\theta_2}{\theta_1}\right)P'_\ell(y)du \\ &\quad + \int_0^1 (1-u)^\ell P'_1\left(1-(1-u)\frac{\theta_2}{\theta_1}\right)P_\ell(y)du\Bigg) \\ &\quad - (\alpha+\beta)(\theta_1-\theta_2)\log T\int_0^1 (1-u)^\ell P'_1\left(1-(1-u)\frac{\theta_2}{\theta_1}\right)P_\ell(y)du.\end{aligned}$$

Since $P_1(0) = P_\ell(0) = 0$ it follows that

$$0 = (1-u)^\ell P_1(u)P_\ell(u)\Big|_{u=0}^{1} = \int_0^1 \Big((1-u)^\ell P_1(u)P_\ell(u)\Big)' du.$$

We can therefore write

$$\ell \int_0^1 (1-u)^{\ell-1} P_1(u)P_\ell(u)du = \int_0^1 (1-u)^\ell P_1\left(1-(1-u)\frac{\theta_2}{\theta_1}\right) P_\ell'(u)du \\ + \int_0^1 (1-u)^\ell P_1'\left(1-(1-u)\frac{\theta_2}{\theta_1}\right) P_\ell(u)du.$$

Combining these observations, we see that

$$I_{12}'(\alpha,\beta) + I_{12}'(-\beta,-\alpha) = \widehat{w}(0)\sum_{\ell=2}^{L}\frac{(-1)^\ell}{(\ell-1)!}\int_0^1 (1-u)^{\ell-1}P_1(u)P_\ell(u)du \\ - \widehat{w}(0)\frac{\theta_1-\theta_2}{\theta_1}\sum_{\ell=2}^{L}\frac{(-1)^\ell}{\ell!}\int_0^1 (1-u)^\ell P_1'\left(1-(1-u)\frac{\theta_2}{\theta_1}\right)P_\ell(u)du.$$

For the expression $(T^{-\alpha-\beta}-1)I_{12}'(-\beta,-\alpha)$, we use (6.18) to find that

$$(T^{-\alpha-\beta}-1)I_{12}'(-\beta,-\alpha) \\ = \frac{\widehat{w}(0)}{\theta_1}\sum_{\ell=2}^{L}\frac{(-1)^\ell}{\ell!}\frac{d^2}{dxdy}\Bigg[y_1^{-\beta x}y_2^{-\alpha y}\int_0^1\int_0^1 T^{-v(\alpha+\beta)}(1-u)^\ell \\ \times P_1\left(x+1-(1-u)\frac{\theta_2}{\theta_1}\right)P_\ell(y+u)dudv\Big|_{x=y=0}\Bigg] \\ + O(T/L).$$

By using similar arguments for the holomorphy of the error terms as in the Section 6.3.1, this completes the proof of Lemma 6.1.2.

6.3.3 The mean value integral $I_{22}(\alpha,\beta)$

This is the hardest case. Once again, we insert the definitions of the Feng mollifiers ψ_2 in the mean value integral $I_{22}(\alpha,\beta)$, so that

$$I_{22}(\alpha,\beta) = \int_{-\infty}^{\infty} w(t)\zeta(\tfrac12+\alpha+it)\zeta(\tfrac12+\beta-it)\psi_F\overline{\psi_F}(\sigma_0+it)dt \\ = \int_{-\infty}^{\infty} w(t)\zeta(\tfrac12+\alpha+it)\zeta(\tfrac12+\beta-it)\sum_{h_1\leqslant y_F}\frac{\mu(h_1)}{h_1^{1/2+it}} \\ \times \sum_{\ell_1=2}^{K}\sum_{p_1p_2\cdots p_{\ell_1}|h_1}\frac{\log p_1\log p_2\cdots\log p_{\ell_1}}{\log^{\ell_1} y_F}P_{\ell_1}[h_1] \\ \times \sum_{h_2\leqslant y_F}\frac{\mu(h_2)}{h_2^{1/2-it}}\sum_{\ell_2=2}^{K}\sum_{q_1q_2\cdots q_{\ell_2}|h_2}\frac{\log q_1\log q_2\cdots\log q_{\ell_2}}{\log^{\ell_2} y_F}P_{\ell_2}[h_2]dt$$

$$
\begin{aligned}
&= \sum_{h_1,h_2 \leqslant y_F} \frac{\mu(h_1)\mu(h_2)}{\sqrt{h_1 h_2}} \\
&\quad \times \sum_{\ell_1=2}^{K} \sum_{\ell_2=2}^{K} \sum_{p_1 p_2 \cdots p_{\ell_1} | h_1} \sum_{q_1 q_2 \cdots q_{\ell_2} | h_2} \frac{\log p_1 \log p_2 \cdots \log p_{\ell_1} \log q_1 \log q_2 \cdots \log q_{\ell_2}}{\log^{\ell_1+\ell_2} y_F} \\
&\quad \times P_{\ell_1}[h_1] P_{\ell_2}[h_2] \int_{-\infty}^{\infty} w(t) \zeta(\tfrac{1}{2}+\alpha+it)\zeta(\tfrac{1}{2}+\beta-it) \left(\frac{h_1}{h_2}\right)^{-it} dt.
\end{aligned}
$$

We already explained in the computation of $I_{12}(\alpha,\beta)$ how to deal with this integral, namely write $I_{22}(\alpha,\beta) = I'_{22}(\alpha,\beta) + I''_{22}(\alpha,\beta)$, where $I''_{22}(\alpha,\beta)$ can be obtained from I'_{22} by switching α and $-\beta$ and multiplying by

$$
\left(\frac{t}{2\pi}\right)^{-\alpha-\beta} = T^{-\alpha-\beta} + O(L^{-1}).
$$

We now use the Mellin integral representations of the polynomials

$$
P_{\ell_1}[h_1] = \sum_i \frac{b_{i,\ell_1}}{\log^i y_2} (\log(y_2/h_1))^i = \sum_i \frac{b_{i,\ell_1} i!}{\log^i y_2} \frac{1}{2\pi i} \int_{(1)} \left(\frac{y_2}{h_1}\right)^u \frac{du}{u^{i+1}},
$$

and

$$
P_{\ell_2}[h_2] = \sum_j \frac{b_{j,\ell_2}}{\log^j y_2} (\log(y_2/h_2))^j = \sum_j \frac{b_{j,\ell_2} j!}{\log^j y_2} \frac{1}{2\pi i} \int_{(1)} \left(\frac{y_2}{h_2}\right)^s \frac{ds}{s^{j+1}}.
$$

This leaves us with

$$
\begin{aligned}
&I'_{22}(\alpha,\beta) \\
&= \int_{-\infty}^{\infty} w(t) \sum_{\ell_1=2}^{K} \sum_{\ell_2=2}^{K} \sum_{i,j} \frac{b_{i,\ell_1} i!}{\log^{i+j} y_2} \frac{b_{j,\ell_2} j!}{\log^{\ell_1+\ell_2} y_2} \\
&\quad \times \left(\frac{1}{2\pi i}\right)^3 \int_{(1)} \int_{(1)} \int_{(1)} y_2^{s+u} g_{\alpha,\beta}(z,t) \frac{G(z)}{z} \\
&\quad \times \sum_{mh_1=nh_2} \frac{\mu(h_1)\mu(h_2)}{h_1^{1/2+u} h_2^{1/2+s} m^{1/2+\alpha+z} n^{1/2+\beta+z}} \\
&\quad \times \sum_{p_1 p_2 \cdots p_{\ell_1} | h_1} \sum_{q_1 q_2 \cdots q_{\ell 2} | h_2} \log p_1 \log p_2 \cdots \log p_{\ell_1} \log q_1 \log q_2 \cdots \log q_{\ell_1} dz \frac{du}{u^{j+1}} \frac{ds}{u^{i+1}} dt.
\end{aligned}
$$

We now have to compute the arithmetical sum $\sum_{mh_1=nh_2}$ with p-adic analysis. The first step is to consolidate the two sums over primes into a single sum. This is accomplished by the use of Lemma 6.2.3. Let us define

$$
\begin{aligned}
S_{\ell_1,\ell_2,k} = &\sum_{mh_1=nh_2} \frac{\mu(h_1)\mu(h_2)}{h_1^{1/2+u} h_2^{1/2+s} m^{1/2+\alpha+z} n^{1/2+\beta+z}} \\
&\times \sum_{\substack{p_1 p_2 \cdots p_{\ell_1+\ell_2-k} | h_1 h_2 \\ p_1 p_2 \cdots p_k | \gcd(h_1,h_2)}} \log^2 p_1 \cdots \log^2 p_k \log p_{k+1} \cdots \log p_{\ell_1+\ell_2-k}.
\end{aligned} \tag{6.26}
$$

The next step is to swap the order of the sums so that

$$S_{\ell_1,\ell_2,k} = (-1)^{\ell_1+\ell_2} \sum_{\substack{p_i \neq p_j \\ q_i \neq q_j \\ r_i \neq r_j \\ p_i \neq q_i \neq r_i}} \frac{\log^2 p_1 \cdots \log^2 p_k \log q_1 \cdots \log q_{\ell_1-k} \log r_1 \cdots \log r_{\ell_2-k}}{(p_1 \cdots p_k q_1 \cdots q_{\ell_1-k})^{1/2+u} (p_1 \cdots p_k r_1 \cdots r_{\ell_2-k})^{1/2+s}}$$
$$\times \sum_{\substack{m\tilde{h}_1 p_1 \cdots p_k q_1 \cdots q_{\ell_1-k} = n \\ \tilde{h}_2 p_1 \cdots p_k r_1 \cdots r_{\ell_2-k}, (\tilde{h}_1, p_1 \cdots p_k q_1 \cdots q_{\ell_1-k})=1 \\ (\tilde{h}_2, p_1 \cdots p_k r_1 \cdots r_{\ell_2-k})=1}} \frac{\mu(\tilde{h}_1)\mu(\tilde{h}_2)}{(\tilde{h}_1)^{1/2+u} (\tilde{h}_2)^{1/2+s} m^{1/2+\alpha+z} n^{1/2+\beta+z}},$$

by making the changes

$$h_1 = \tilde{h}_1 p_1 \cdots p_k q_1 \cdots q_{\ell_1-k},$$
$$h_2 = \tilde{h}_2 p_1 \cdots p_k r_1 \cdots r_{\ell_2-k},$$

implying that

$$(\tilde{h}_1, p_1 \cdots p_k q_1 \cdots q_{\ell_1-k}) = 1,$$
$$(\tilde{h}_2, p_1 \cdots p_k r_1 \cdots r_{\ell_2-k}) = 1,$$
$$(q_1 \cdots q_{\ell_1-k}, r_1 \cdots r_{\ell_2-k}) = 1,$$

so that

$$S_{\ell_1,\ell_2,k} = (-1)^{\ell_1+\ell_2} \sum_{\substack{p_i \neq p_j \\ q_i \neq q_j \\ r_i \neq r_j \\ p_i \neq q_i \neq r_i}} \frac{\log^2 p_1 \cdots \log^2 p_k \log q_1 \cdots \log q_{\ell_1-k} \log r_1 \cdots \log r_{\ell_2-k}}{(p_1 \cdots p_k q_1 \cdots q_{\ell_1-k})^{1/2+u} (p_1 \cdots p_k r_1 \cdots r_{\ell_2-k})^{1/2+s}}$$
$$\times \sum_{\substack{m\tilde{h}_1 p_1 \cdots p_k q_1 \cdots q_{\ell_1-k} = n \\ \tilde{h}_2 p_1 \cdots p_k r_1 \cdots r_{\ell_2-k}, (\tilde{h}_1, p_1 \cdots p_k q_1 \cdots q_{\ell_1-k})=1 \\ (\tilde{h}_2, p_1 \cdots p_k r_1 \cdots r_{\ell_2-k})=1}} \frac{\mu(\tilde{h}_1)\mu(\tilde{h}_2)}{(\tilde{h}_1)^{1/2+u} (\tilde{h}_2)^{1/2+s} m^{1/2+\alpha+z} n^{1/2+\beta+z}}.$$

Here the p's, the q's and the r's are all primes. Let us define the inner sum to be $\tilde{S}_{\ell_1,\ell_2,k}$ and let us recall that $\nu_p(n) = n'$ is the number of times the prime p appears in n so that

$$\tilde{S}_{\ell_1,\ell_2,k} = \sum_{\substack{m\tilde{h}_1 p_1 \cdots p_k q_1 \cdots q_{\ell_1-k} = n \\ \tilde{h}_2 p_1 \cdots p_k r_1 \cdots r_{\ell_2-k}, (\tilde{h}_1, p_1 \cdots p_k q_1 \cdots q_{\ell_1-k})=1 \\ (\tilde{h}_2, p_1 \cdots p_k r_1 \cdots r_{\ell_2-k})=1}} \frac{\mu(\tilde{h}_1)\mu(\tilde{h}_2)}{(\tilde{h}_1)^{1/2+u} (\tilde{h}_2)^{1/2+s} m^{1/2+\alpha+z} n^{1/2+\beta+z}}$$
$$= \prod_{p \in \{p_1, \cdots, p_k\}} \sum_{n'=m'} \frac{1}{(p^{m'})^{1/2+\alpha+z} (p^{n'})^{1/2+\beta+z}}$$
$$\times \prod_{q \in \{q_1, \cdots, q_{\ell_1-k}\}} \sum_{1+m'=n'+\tilde{h}_2'} \frac{\mu(q^{\tilde{h}_2'})}{(q^{\tilde{h}_2'})^{1/2+s} (q^{m'})^{1/2+\alpha+z} (q^{n'})^{1/2+\beta+z}}$$

$$
\begin{aligned}
&\times \prod_{r\in\{r_1,\cdots,r_{\ell_2-k}\}} \sum_{\tilde{h}'_1+m'=n'+1} \frac{\mu(r^{\tilde{h}'_1})}{(r^{\tilde{h}'_1})^{1/2+u}(r^{m'})^{1/2+\alpha+z}(r^{n'})^{1/2+\beta+z}} \\
&\times \prod_{p\notin\{p_1,\cdots,p_k\}\cup\{q_1,\cdots,q_{\ell_1-k}\}\cup\{r_1,\cdots,r_{\ell_2-k}\}} \\
&\times \sum_{\tilde{h}'_1+m'=n'+\tilde{h}'_2} \frac{\mu(p^{\tilde{h}'_1})\mu(p^{\tilde{h}'_2})}{(p^{\tilde{h}'_1})^{1/2+u}(p^{\tilde{h}'_2})^{1/2+s}(p^{m'})^{1/2+\alpha+z}(p^{n'})^{1/2+\beta+z}} \\
= {}& \frac{\Pi_1(\alpha,\beta,s,u,z)\Pi_2(\alpha,\beta,s,u,z)\Pi_3(\alpha,\beta,s,u,z)\Pi_4(\alpha,\beta,s,u,z)}{\Pi_5(\alpha,\beta,s,u,z)}.
\end{aligned}
$$

Each product is evaluated by

$$
\begin{aligned}
\Pi_1(\alpha,\beta,s,u,z) &= \prod_p \sum_{\tilde{h}'_1+m'=n'+\tilde{h}'_2} \frac{\mu(p^{\tilde{h}'_1})\mu(p^{\tilde{h}'_2})}{(p^{\tilde{h}'_1})^{1/2+u}(p^{\tilde{h}'_2})^{1/2+s}(p^{m'})^{1/2+\alpha+z}(p^{n'})^{1/2+\beta+z}} \\
&= \prod_p \left(1+\frac{1}{p^{1+s+u}}-\frac{1}{p^{1+s+\alpha+z}}-\frac{1}{p^{1+u+\beta+z}}+\frac{1}{p^{1+\alpha+\beta+2z}}+O(p^{-2+\varepsilon})\right),
\end{aligned}
$$

then

$$
\begin{aligned}
\Pi_2(\alpha,\beta,s,u,z) &= \prod_{p\in\{p_1,\cdots,p_k\}} \sum_{n'=m'} \frac{1}{(p^{m'})^{1/2+\alpha+z}(p^{n'})^{1/2+\beta+z}} \\
&= \prod_{p\in\{p_1,\cdots,p_k\}} \left(1+\frac{1}{p^{1+\alpha+\beta+2z}}+O(p^{-2+\varepsilon})\right). \qquad (6.27)
\end{aligned}
$$

It is followed by

$$
\begin{aligned}
\Pi_3(\alpha,\beta,s,u,z) &= \prod_{q\in\{q_1,\cdots,q_{\ell_1-k}\}} \sum_{m'+1=n'+\tilde{h}'_2} \frac{\mu(q^{\tilde{h}'_2})}{(q^{\tilde{h}'_2})^{1/2+s}(q^{m'})^{1/2+\alpha+z}(q^{n'})^{1/2+\beta+z}} \\
&= \prod_{q\in\{q_1,\cdots,q_{\ell_1-k}\}} \left(-\frac{1}{q^{1/2+s}}+\frac{1}{q^{1/2+\beta+z}}+O(q^{-2+\varepsilon})\right),
\end{aligned}
$$

as well as

$$
\begin{aligned}
\Pi_4(\alpha,\beta,s,u,z) &= \prod_{r\in\{r_1,\cdots,r_{\ell_2-k}\}} \sum_{\tilde{h}'_1+m'=n'+1} \frac{\mu(r^{\tilde{h}'_1})}{(r^{\tilde{h}'_1})^{1/2+u}(r^{m'})^{1/2+\alpha+z}(r^{n'})^{1/2+\beta+z}} \\
&= \prod_{r\in\{r_1,\cdots,r_{\ell_2-k}\}} \left(-\frac{1}{r^{1/2+u}}+\frac{1}{r^{1/2+\alpha+z}}+O(r^{-2+\varepsilon})\right),
\end{aligned}
$$

and finally

$$
\begin{aligned}
\Pi_5(\alpha,\beta,s,u,z) = {}& \prod_{p\in\{p_1,\cdots,p_k\}\cup\{q_1,\cdots,q_{\ell_1-k}\}\cup\{r_1,\cdots,r_{\ell_2-k}\}} \\
&\times \sum_{\tilde{h}'_1+m'=n'+\tilde{h}'_2} \frac{\mu(p^{\tilde{h}'_1})\mu(p^{\tilde{h}'_2})}{(p^{\tilde{h}'_1})^{1/2+u}(p^{\tilde{h}'_2})^{1/2+s}(p^{m'})^{1/2+\alpha+z}(p^{n'})^{1/2+\beta+z}}
\end{aligned}
$$

$$= \prod_{p\in\{p_1,\cdots,p_k\}\cup\{q_1,\cdots,q_{\ell_1-k}\}\cup\{r_1,\cdots,r_{\ell_2-k}\}} \times\left(1+\frac{1}{p^{1+s+u}}-\frac{1}{p^{1+s+\alpha+z}}-\frac{1}{p^{1+u+\beta+z}}+\frac{1}{p^{1+\alpha+\beta+2z}}+O(p^{-2+\varepsilon})\right).$$

This leaves us with

$$\begin{aligned}\tilde{S}_{\ell_1,\ell_2,k} &= \prod_p\left(1+\frac{1}{p^{1+s+u}}-\frac{1}{p^{1+s+\alpha+z}}-\frac{1}{p^{1+u+\beta+z}}+\frac{1}{p^{1+\alpha+\beta+2z}}+O(p^{-2+\varepsilon})\right)\\ &= \frac{\zeta(1+s+u)\zeta(1+\alpha+\beta+2z)}{\zeta(1+s+\alpha+z)\zeta(1+u+\beta+z)}A_{\alpha,\beta}(s,u,z),\end{aligned}$$

where A is an arithmetical factor that is given by an absolutely convergent Euler product in some product of half-planes containing the origin. From our previous analysis of the $I_{12}(\alpha,\beta)$ case, we know that $A_{0,0}(z,z,z)=1$ for all values of z. Therefore we end up with

$$\begin{aligned}S_{\ell_1,\ell_2,k} &= \frac{\zeta(1+s+u)\zeta(1+\alpha+\beta+2z)}{\zeta(1+s+\alpha+z)\zeta(1+u+\beta+z)}A_{\alpha,\beta}(s,u,z)\\ &\times(-1)^{\ell_1+\ell_2}\sum_{\substack{p_i\neq p_j\\ q_i\neq q_j\\ r_i\neq r_j\\ p_i\neq q_i\neq r_i}}\log^2 p_1\cdots\log^2 p_k\log q_1\cdots\log q_{\ell_1-k}\log r_1\cdots\log r_{\ell_2-k}\\ &\times\prod_{p\in\{p_1,\cdots,p_k\}}\frac{E_1(p)+O(p^{-2+\varepsilon})}{1+\frac{1}{p^{1+s+\alpha+z}}+\frac{1}{p^{1+u+\beta+z}}-\frac{1}{p^{1+\alpha+\beta+2z}}+E_1(p)+O(p^{-2+\varepsilon})}\\ &\times\prod_{q\in\{q_1,\cdots,q_{\ell_1-k}\}}\frac{E_2(q)+O(q^{-2+\varepsilon})}{1+\frac{1}{q^{1+s+\alpha+z}}-\frac{1}{q^{1+\alpha+\beta+2z}}-E_2(q)+O(q^{-2+\varepsilon})}\\ &\times\prod_{r\in\{r_1,\cdots,r_{\ell_2-k}\}}\frac{E_3(r)+O(r^{-2+\varepsilon})}{1+\frac{1}{r^{1+u+\beta+z}}-\frac{1}{r^{1+\alpha+\beta+2z}}-E_3(r)+O(r^{-2+\varepsilon})},\end{aligned}$$

where

$$E_1(p)=\frac{1}{p^{1+s+u}},$$

and

$$E_2(q)=\frac{1}{q^{1/2+u}}\left(\frac{1}{q^{1/2+s}}-\frac{1}{q^{1/2+\beta+z}}\right)=\frac{1}{q^{1+s+u}}-\frac{1}{q^{1+\beta+u+z}},$$

and finally

$$E_3(r)=\frac{1}{r^{1/2+s}}\left(\frac{1}{q^{1/2+u}}-\frac{1}{q^{1/2+\alpha+z}}\right)=\frac{1}{r^{1+s+u}}-\frac{1}{r^{1+\alpha+s+z}}.$$

We define $H_{\ell_1,\ell_2,k}$ to be the last part of $S_{\ell_1,\ell_2,k}$. This means that

$$\begin{aligned}&H_{\ell_1,\ell_2,k}\\ &=(-1)^{\ell_1+\ell_2}\sum_{\substack{p_i\neq p_j\\ q_i\neq q_j\\ r_i\neq r_j\\ p_i\neq q_i\neq r_i}}(E_1(p)+O(p^{-2+\varepsilon}))\log^2 p\end{aligned}$$

$$
\begin{aligned}
&\times \left(1 - E_1(p) - \frac{1}{p^{1+s+\alpha+z}} - \frac{1}{p^{1+u+\beta+z}} + \frac{1}{p^{1+\alpha+\beta+2z}} + O(p^{-2+\varepsilon})\right) \\
&\times (E_2(q) + O(q^{-2+\varepsilon})) \log q \left(1 + E_2(q) - \frac{1}{q^{1+s+\alpha+z}} + \frac{1}{q^{1+\alpha+\beta+2z}} + O(q^{-2+\varepsilon})\right) \\
&\times (E_3(r) + O(r^{-2+\varepsilon})) \log r \left(1 + E_3(r) - \frac{1}{r^{1+u+\beta+z}} + \frac{1}{r^{1+\alpha+\beta+2z}} + O(r^{-2+\varepsilon})\right) \\
= {}& (-1)^{\ell_1+\ell_2} \sum_{\substack{p_i \neq p_j \\ q_i \neq q_j \\ r_i \neq r_j \\ p_i \neq q_i \neq r_i}} \prod_{p \in \{p_1, \cdots, p_k\}} \left(E_1(p) \log^2 p + O\left(\frac{\log^2 p}{p^{2-\varepsilon}}\right)\right) \\
&\times \prod_{q \in \{q_1, \cdots, q_{\ell_1-k}\}} \left(E_2(q) \log q + O\left(\frac{\log q}{q^{2-\varepsilon}}\right)\right) \prod_{r \in \{r_1, \cdots, r_{\ell_2-k}\}} \left(E_3(r) \log r + O\left(\frac{\log r}{r^{2-\varepsilon}}\right)\right) \\
&+ O(f(p^{-2+\varepsilon}, q^{-2+\varepsilon}, r^{-2+\varepsilon})),
\end{aligned}
$$

for some polynomial f. Applying the inclusion-exclusion principle we then have

$$
\begin{aligned}
H_{\ell_1,\ell_2,k} = (-1)^{\ell_1+\ell_2} & \left(\sum_p E_1(p) \log^2 p + O\left(\frac{\log^2 p}{p^{2-\varepsilon}}\right)\right)^k \\
& \times \left(\sum_q E_2(q) \log q + O\left(\frac{\log q}{q^{2-\varepsilon}}\right)\right)^{\ell_1-k} \left(\sum_r E_3(r) \log r + O\left(\frac{\log r}{r^{2-\varepsilon}}\right)\right)^{\ell_2-k} \\
& + \sum_{p,q,r} B(p,q,r),
\end{aligned}
$$

where

$$
B(p,q,r) \ll_{\alpha,\beta,s,u,z,\varepsilon} f\left(\frac{1}{p^{2-\varepsilon}}, \frac{1}{q^{2-\varepsilon}}, \frac{1}{r^{2-\varepsilon}}\right).
$$

As in the previous crossterm, we now need to identify the logarithms of the primes with the signature of the von Mangoldt functions $\Lambda(n)$ and $\Lambda_2(n)$. With this in mind, we first write

$$
\frac{\zeta'}{\zeta}(s) = -\sum_{n=1}^{\infty} \Lambda(n) n^{-s} = -\sum_p \frac{\log p}{p^s}\left(1 - \frac{1}{p^s}\right)^{-1} = -\sum_p \frac{\log p}{p^s} + O\left(\frac{\log p}{p^{2s}}\right),
$$

and

$$
\frac{\zeta''}{\zeta}(s) = \sum_{n=1}^{\infty} \Lambda_2(n) n^{-s} = \sum_p \frac{\log^2 p}{p^s}\left(1 - \frac{1}{p^s}\right)^{-1} = \sum_p \frac{\log^2 p}{p^s} + O\left(\frac{\log^2 p}{p^{2s}}\right),
$$

for $\mathrm{Re}(s) > 1$. This means that

$$
\begin{aligned}
H_{\ell_1,\ell_2,k} &= (-1)^{\ell_1+\ell_2} \left(\frac{\zeta''}{\zeta}(1+s+u)\right)^k \left(-\frac{\zeta'}{\zeta}(1+s+u) + \frac{\zeta'}{\zeta}(1+\beta+u+z)\right)^{\ell_1-k} \\
&\quad \times \left(-\frac{\zeta'}{\zeta}(1+s+u) + \frac{\zeta'}{\zeta}(1+\alpha+s+z)\right)^{\ell_2-k} + D(\alpha,\beta,s,u,z) \\
&= (-1)^{\ell_1+\ell_2} (-V_1)^k (-V_2)^{\ell_1-k} (-V_3)^{\ell_2-k}
\end{aligned}
$$

$$+\sum_{l=0}^{k-1} V_1^l A_l(\alpha,\beta,s,u,z) \sum_{m=0}^{\ell_1-k-1} V_2^m B_m(\alpha,\beta,s,u,z) \sum_{n=0}^{\ell_2-k-1} V_3^n C_n(\alpha,\beta,s,u,z),$$

where $D(\alpha,\beta,s,u,z)$ are terms of smaller order and where

$$V_1 = -\frac{\zeta''}{\zeta}(1+s+u),\ V_2 = \frac{\zeta'}{\zeta}(1+s+u) - \frac{\zeta'}{\zeta}(1+\beta+u+z),\ V_3 = \frac{\zeta'}{\zeta}(1+s+u) - \frac{\zeta'}{\zeta}(1+\alpha+s+z).$$

Moreover,

$$A_l(\alpha,\beta,s,u,z) \ll_{\alpha,\beta,s,u,z,\varepsilon} \sum_p \frac{\log^2 p}{p^{2-\varepsilon}},$$
$$B_m(\alpha,\beta,s,u,z), C_n(\alpha,\beta,s,u,z) \ll_{\alpha,\beta,s,u,z,\varepsilon} \sum_p \frac{\log p}{p^{2-\varepsilon}}.$$

All of these terms are analytic in a larger region of the complex plane, thus we are only interested in the term $(-V_1)^k(-V_2)^{\ell_1-k}(-V_3)^{\ell_2-k}$. Consequently, the end result of this computation is that

$$\begin{aligned}
&I'_{22}(\alpha,\beta)\\
&= \int_{-\infty}^{\infty} w(t) \sum_{\ell_1=2}^{K}\sum_{\ell_2=2}^{K}\sum_{i,j}\sum_{k=0}^{\min(\ell_1,\ell_2)} \binom{\ell_1}{k}(\ell_2)_k \frac{b_{i,\ell_1} i!}{\log^{i+j} y_2}\frac{b_{j,\ell_2} j!}{\log^{\ell_1+\ell_2} y_2}\\
&\times \left(\frac{1}{2\pi i}\right)^3 \int_{(1)}\int_{(1)}\int_{(1)} y_2^{s+u} g_{\alpha,\beta}(z,t)\frac{G(z)}{z}\frac{\zeta(1+s+u)\zeta(1+\alpha+\beta+2z)}{\zeta(1+s+\alpha+z)\zeta(1+u+\beta+z)}A_{\alpha,\beta}(s,u,z)\\
&\times (-1)^{\ell_1+\ell_2}\left(\frac{\zeta''}{\zeta}(1+s+u)\right)^k \left(-\frac{\zeta'}{\zeta}(1+s+u)+\frac{\zeta'}{\zeta}(1+\beta+u+z)\right)^{\ell_1-k}\\
&\times \left(-\frac{\zeta'}{\zeta}(1+s+u)+\frac{\zeta'}{\zeta}(1+\alpha+s+z)\right)^{\ell_2-k} dz\frac{du}{u^{i+1}}\frac{ds}{s^{j+1}}dt.
\end{aligned}$$

We now take the s,u,z contours of integration to $\delta>0$ small, and then move z to $-\delta+\varepsilon$, crossing a simple pole at $z=0$ only (since, yet again, $G(z)$ vanishes at the pole of $\zeta(1+\alpha+\beta+2z)$). The new line of integration I''_{22} contributes

$$I''_{22} \ll T^{1+\varepsilon}\left(\frac{y_2^2}{T}\right)^{\delta} \ll T^{1-\varepsilon},$$

since $\theta_2 = 1/2-\varepsilon$. Write $I'_{22}(\alpha,\beta) = I'_{220}(\alpha,\beta)+O(T^{1-\varepsilon})$, where $I'_{220}(\alpha,\beta)$ corresponds to the residue at $z=0$, i.e.

$$\begin{aligned}
&I'_{220}(\alpha,\beta)\\
&= \int_{-\infty}^{\infty} w(t) \sum_{\ell_1=2}^{K}\sum_{\ell_2=2}^{K}\sum_{i,j}\sum_{k=0}^{\min(\ell_1,\ell_2)} \binom{\ell_1}{k}(\ell_2)_k \frac{b_{i,\ell_1} i!}{\log^{i+j} y_2}\frac{b_{j,\ell_2} j!}{\log^{\ell_1+\ell_2} y_2}\\
&\times \left(\frac{1}{2\pi i}\right)^2 \int_{(\delta)}\int_{(\delta)} \operatorname*{Res}_{z=0}\frac{G(z)}{z} g_{\alpha,\beta}(z,t) y_2^{s+u}\frac{\zeta(1+s+u)\zeta(1+\alpha+\beta+2z)}{\zeta(1+s+\alpha+z)\zeta(1+u+\beta+z)}A_{\alpha,\beta}(s,u,z)\\
&\times (-1)^{\ell_1+\ell_2}\left(\frac{\zeta''}{\zeta}(1+s+u)\right)^k \left(-\frac{\zeta'}{\zeta}(1+s+u)+\frac{\zeta'}{\zeta}(1+\beta+u+z)\right)^{\ell_1-k}
\end{aligned}$$

$$\times\left(-\frac{\zeta'}{\zeta}(1+s+u)+\frac{\zeta'}{\zeta}(1+\alpha+s+z)\right)^{\ell_2-k}\frac{du}{u^{i+1}}\frac{ds}{s^{j+1}}dt$$
$$=\widehat{w}(0)\zeta(1+\alpha+\beta)\sum_{\ell_1=2}^{K}\sum_{\ell_2=2}^{K}\sum_{i,j}\sum_{k=0}^{\min(\ell_1,\ell_2)}\binom{\ell_1}{k}(\ell_2)_k(-1)^{\ell_1+\ell_2}\frac{b_{i,\ell_1}i!}{\log^{i+j}y_2}\frac{b_{j,\ell_2}j!}{\log^{\ell_1+\ell_2}y_2}J_{22},$$

where

$$J_{22}=\left(\frac{1}{2\pi i}\right)^2\int_{(\delta)}\int_{(\delta)}y_2^{s+u}\frac{\zeta(1+s+u)}{\zeta(1+s+\alpha)\zeta(1+u+\beta)}A_{\alpha,\beta}(s,u,0)$$
$$\times\left(\frac{\zeta''}{\zeta}(1+s+u)\right)^k\left(-\frac{\zeta'}{\zeta}(1+s+u)+\frac{\zeta'}{\zeta}(1+\beta+u)\right)^{\ell_1-k}$$
$$\times\left(-\frac{\zeta'}{\zeta}(1+s+u)+\frac{\zeta'}{\zeta}(1+\alpha+s)\right)^{\ell_2-k}\frac{du}{u^{i+1}}\frac{ds}{s^{j+1}}.$$

The next step is to employ the binomial theorem in the part of the integrand that involves ζ functions. Calling this part $\mathcal{Z}$, we then have

$$\mathcal{Z}(s,u)$$
$$:=\frac{\zeta(1+s+u)}{\zeta(1+s+\alpha)\zeta(1+u+\beta)}\left(\frac{\zeta''}{\zeta}(1+s+u)\right)^k$$
$$\times\left(-\frac{\zeta'}{\zeta}(1+s+u)+\frac{\zeta'}{\zeta}(1+\beta+u)\right)^{\ell_1-k}\left(-\frac{\zeta'}{\zeta}(1+s+u)+\frac{\zeta'}{\zeta}(1+\alpha+s)\right)^{\ell_2-k}$$
$$=\frac{\zeta(1+s+u)}{\zeta(1+s+\alpha)\zeta(1+u+\beta)}\left(\frac{\zeta''}{\zeta}(1+s+u)\right)^k$$
$$\times\sum_{r_1=0}^{\ell_1-k}\binom{\ell_1-k}{r_1}\left(\frac{\zeta'}{\zeta}(1+\beta+u)\right)^{\ell_1-k-r_1}\left(-\frac{\zeta'}{\zeta}(1+s+u)\right)^{r_1}$$
$$\times\sum_{r_2=0}^{\ell_2-k}\binom{\ell_2-k}{r_2}\left(\frac{\zeta'}{\zeta}(1+\alpha+s)\right)^{\ell_2-k-r_2}\left(-\frac{\zeta'}{\zeta}(1+s+u)\right)^{r_2}$$
$$=\sum_{r_1=0}^{\ell_1-k}\sum_{r_2=0}^{\ell_2-k}\binom{\ell_1-k}{r_1}\binom{\ell_2-k}{r_2}\sum_{n=1}^{\infty}\frac{(\mathbf{1}*\Lambda_2^{*k}*\Lambda^{*r_1+r_2})(n)}{n^{1+s+u}}$$
$$\times\frac{1}{\zeta(1+\beta+u)}\left(\frac{\zeta'}{\zeta}(1+\beta+u)\right)^{\ell_1-k-r_1}\frac{1}{\zeta(1+s+\alpha)}\left(\frac{\zeta'}{\zeta}(1+\alpha+s)\right)^{\ell_2-k-r_2},$$

where we have used the Dirichlet convolution of

$$\zeta(s)=\sum_{n=1}^{\infty}\frac{1}{n^s},\quad\frac{\zeta'}{\zeta}(s)=-\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s},\quad\text{and}\quad\frac{\zeta''}{\zeta}(s)=\sum_{n=1}^{\infty}\frac{\Lambda_2(n)}{n^s},$$

for $\mathrm{Re}(s)>1$. Now we take $\delta\asymp L^{-1}$ and bound the integral trivially to get $J_{22}\ll L^{i+j-1}$. This means that we can use a Taylor series expansion so that $A_{\alpha,\beta}(s,u,0)=A_{0,0}(0,0,0)+O(|s|+|u|)$ to write $J_{22}(\alpha,\beta)=J_{22}'(\alpha,\beta)+O(L^{i+j-2})$, say. We recall that earlier we proved that $A_{0,0}(z,z,z)=1$ for all z, and hence $A_{0,0}(0,0,0)=1$. This has the effect of separating the complex variables s and u as follows:

$$J'_{22} = \sum_{n \leqslant y_2} \sum_{r_1=0}^{\ell_1-k} \sum_{r_2=0}^{\ell_2-k} \binom{\ell_1-k}{r_1}\binom{\ell_2-k}{r_2} \frac{(\mathbf{1} * \Lambda_2^{*k} * \Lambda^{*r_1+r_2})(n)}{n} L_{22,1} L_{22,2},$$

where

$$L_{22,1} = \frac{1}{2\pi i} \int_{(\delta)} \left(\frac{y_2}{n}\right)^s \frac{1}{\zeta(1+s+\alpha)} \left(\frac{\zeta'}{\zeta}(1+\alpha+s)\right)^{\ell_2-k-r_2} \frac{ds}{s^{j+1}}, \tag{6.28}$$

and

$$L_{22,2} = \frac{1}{2\pi i} \int_{(\delta)} \left(\frac{y_2}{n}\right)^u \frac{1}{\zeta(1+\beta+u)} \left(\frac{\zeta'}{\zeta}(1+\beta+u)\right)^{\ell_1-k-r_1} \frac{du}{u^{i+1}}.$$

These two integrals are identical, up to the symmetries in s/u, ℓ_1/ℓ_2, α/β and r_1/r_2 and they were in fact treated in the $I_{12}(\alpha, \beta)$ case. The end results for the main terms are

$$\begin{aligned} &L_{22,1} \\ &= \frac{1}{2\pi i} \oint \left(\frac{y_2}{n}\right)^s (s+\alpha)^{1-\ell_2+k+r_2} \frac{ds}{s^{j+1}} = \frac{(-1)^{\ell_2-k+r_2}}{j!} \frac{d^{1-\ell_2+k+r_2}}{dx^{1-\ell_2+k+r_2}} e^{\alpha x} \left(x + \log\frac{y_2}{n}\right)^j \Big|_{x=0}, \end{aligned}$$

and

$$\begin{aligned} &L_{22,2} \\ &= \frac{1}{2\pi i} \oint \left(\frac{y_2}{n}\right)^u (u+\beta)^{1-\ell_1+k+r_1} \frac{du}{u^{i+1}} = \frac{(-1)^{\ell_1-k+r_1}}{i!} \frac{d^{1-\ell_1+k+r_1}}{dy^{1-\ell_1+k+r_1}} e^{\beta y} \left(y + \log\frac{y_2}{n}\right)^i \Big|_{y=0}. \end{aligned}$$

Next, we insert these results into J'_{22} and we end up with

$$\begin{aligned} J'_{22} = \frac{1}{i!}\frac{1}{j!} &\sum_{n \leqslant y_2} \sum_{r_1=0}^{\ell_1-k} \sum_{r_2=0}^{\ell_2-k} (-1)^{\ell_1+\ell_2-2k+r_1+r_2} \binom{\ell_1-k}{r_1}\binom{\ell_2-k}{r_2} \frac{(\mathbf{1} * \Lambda_2^{*k} * \Lambda^{*r_1+r_2})(n)}{n} \\ &\times \frac{d^{1-\ell_2+k+r_2}}{dx^{1-\ell_2+k+-r_2}} \frac{d^{1-\ell_1+k+r_1}}{dy^{1-\ell_1+k+r_1}} e^{\alpha x+\beta y} \left(x + \log\frac{y_2}{n}\right)^j \Big|_{x=0} \left(y + \log\frac{y_2}{n}\right)^i \Big|_{y=0} \\ &+ O(L^{i+j-2}). \end{aligned}$$

To make matters easier, we again employ the change of variables

$$x \to \frac{x}{\log y_2} \quad \text{and} \quad y \to \frac{y}{\log y_2},$$

and this produces

$$\begin{aligned} &J'_{22} \\ &= \frac{\log^{i+j-2} y_2}{i!j!} \sum_{n \leqslant y_2} \sum_{r_1=0}^{\ell_1-k} \sum_{r_2=0}^{\ell_2-k} (-1)^{\ell_1+\ell_2-2k+r_1+r_2} \binom{\ell_1-k}{r_1}\binom{\ell_2-k}{r_2} \frac{(\mathbf{1} * \Lambda_2^{*k} * \Lambda^{*r_1+r_2})(n)}{n} \\ &\times \frac{d^{1-\ell_2+k+r_2}}{dx^{1-\ell_2+k+r_2}} \frac{d^{1-\ell_1+k+r_1}}{dy^{1-\ell_1+k+r_1}} y_2^{\alpha x+\beta y} \left(x + \frac{\log(y_2/n)}{\log y_2}\right)^j \Big|_{x=0} \left(y + \frac{\log(y_2/n)}{\log y_2}\right)^i \Big|_{y=0} \\ &+ O(L^{i+j-2}). \end{aligned}$$

We are now ready to insert this into I'_{220} so that

$$\begin{aligned}
&I'_{220}(\alpha,\beta)\\
&=\frac{\widehat{w}(0)}{(\alpha+\beta)\log^2 y_2}\frac{d^2}{dxdy}\Bigg[y_2^{\alpha x+\beta y}\sum_{\ell_1=2}^{K}\sum_{\ell_2=2}^{K}\frac{1}{\log^{\ell_1+\ell_2}y_2}\sum_{i,j}\sum_{k=0}^{\min(\ell_1,\ell_2)}\binom{\ell_1}{k}(\ell_2)_k a_{i,\ell_1}a_{j,\ell_2}\\
&\quad\times\sum_{n\leqslant y_2}\sum_{r_1=0}^{\ell_1-k}\sum_{r_2=0}^{\ell_2-k}(-1)^{r_1+r_2}\binom{\ell_1-k}{r_1}\binom{\ell_2-k}{r_2}\frac{(\mathbf{1}*\Lambda_2^{*k}*\Lambda^{*r_1+r_2})(n)}{n}\\
&\quad\times\frac{d^{k-\ell_2+r_2}}{dx^{k-\ell_2+r_2}}\frac{d^{k-\ell_1+r_1}}{dy^{k-\ell_1+r_1}}\left(x+\frac{\log(y_2/n)}{\log y_2}\right)^j\Bigg|_{x=0}\left(y+\frac{\log(y_2/n)}{\log y_2}\right)^i\Bigg|_{y=0}\Bigg]\\
&\quad+O(T/L),
\end{aligned}$$

where we have used $\zeta(1+\alpha+\beta)=1/(\alpha+\beta)+O(1)$. We now sum over i and j, e.g.

$$P_{\ell_1}\left(x+\frac{\log(y_2/n)}{\log y_2}\right)=\sum_i b_{i,\ell_1}\left(x+\frac{\log(y_2/n)}{\log y_2}\right)^i,$$

thereby obtaining

$$\begin{aligned}
I'_{220}(\alpha,\beta)&=\frac{\widehat{w}(0)}{(\alpha+\beta)\log^2 y_2}\frac{d^2}{dxdy}\Bigg[y_2^{\alpha x+\beta y}\sum_{\ell_1=2}^{K}\sum_{\ell_2=2}^{K}\frac{1}{\log^{\ell_1+\ell_2}y_2}\sum_{k=0}^{\min(\ell_1,\ell_2)}\binom{\ell_1}{k}(\ell_2)_k\\
&\quad\times\sum_{n\leqslant y_2}\sum_{r_1=0}^{\ell_1-k}\sum_{r_2=0}^{\ell_2-k}(-1)^{r_1+r_2}\binom{\ell_1-k}{r_1}\binom{\ell_2-k}{r_2}\frac{(\mathbf{1}*\Lambda_2^{*k}*\Lambda^{*r_1+r_2})(n)}{n}\\
&\quad\times\frac{d^{k-\ell_2+r_2}}{dx^{k-\ell_2+r_2}}\frac{d^{k-\ell_1+r_1}}{dy^{k-\ell_1+r_1}}P_{\ell_1}\left(x+\frac{\log(y_2/n)}{\log y_2}\right)\Bigg|_{x=0}P_{\ell_2}\left(y+\frac{\log(y_2/n)}{\log y_2}\right)\Bigg|_{y=0}\Bigg]\\
&\quad+O(T/L).
\end{aligned}$$

Lemma 6.2.7 gives us

$$\begin{aligned}
&\sum_{n\leqslant y_2}\frac{(\mathbf{1}*\Lambda_2^{*k}*\Lambda^{*r_1+r_2})(n)}{n}P_{\ell_1}\left(x+\frac{\log(y_2/n)}{\log y_2}\right)P_{\ell_2}\left(y+\frac{\log(y_2/n)}{\log y_2}\right)\\
&=\frac{2^{r_1+r_2}\log^{1+2k+r_1+r_2}y_2}{(1+r_1+r_2+2k)!}\int_0^1(1-u)^{2k+r_1+r_2}P_{\ell_1}(x+u)P_{\ell_2}(y+u)du+O(\log^{2k+r_1+r_2}y_2),
\end{aligned}$$

so that we we are left with

$$\begin{aligned}
I'_{220}(\alpha,\beta)&=\frac{\widehat{w}(0)}{(\alpha+\beta)\log^2 y_2}\frac{d^2}{dxdy}\Bigg[y_2^{\alpha x+\beta y}\sum_{\ell_1=2}^{K}\sum_{\ell_2=2}^{K}\frac{1}{\log^{\ell_1+\ell_2}y_2}\sum_{k=0}^{\min(\ell_1,\ell_2)}\binom{\ell_1}{k}(\ell_2)_k\\
&\quad\times\sum_{r_1=0}^{\ell_1}\sum_{r_2=0}^{\ell_2}(-1)^{r_1+r_2}\binom{\ell_1-k}{r_1}\binom{\ell_2-k}{r_2}\frac{d^{k-\ell_2+r_2}}{dx^{k-\ell_2+r_2}}\frac{d^{k-\ell_1+r_1}}{dy^{k-\ell_1+r_1}}\\
&\quad\times\frac{2^{r_1+r_2}\log^{1+r_1+r_2+2k}y_2}{(1+r_1+r_2+2k)!}\int_0^1(1-u)^{2k+r_1+r_2}P_{\ell_1}(x+u)P_{\ell_2}(y+u)du\Bigg|_{x=y=0}\Bigg]\\
&\quad+O(T/L).
\end{aligned}$$

Note that $r_1 \le \ell_1 - k$ and $r_2 \le \ell_2 - k$. Thus only the cases $r_1 = \ell_1 - k$ and $r_2 = \ell_2 - k$ contribute to the main term. We therefore have

$$I'_{220}(\alpha,\beta) = \frac{\widehat{w}(0)}{(\alpha+\beta)\log y_2} \frac{d^2}{dxdy}\Bigg[y_2^{\alpha x+\beta y} \sum_{\ell_1=2}^{K}\sum_{\ell_2=2}^{K}\sum_{k=0}^{\min(\ell_1,\ell_2)} \binom{\ell_1}{k}(\ell_2)_k(-1)^{\ell_1+\ell_2-2k} \\ \times \frac{2^{\ell_1+\ell_2-2k}}{(\ell_1+\ell_2)!}\int_0^1 (1-u)^{\ell_1+\ell_2} P_{\ell_1}(x+u)P_{\ell_2}(y+u)du\Big|_{x=y=0}\Bigg] + O(T/L).$$

Recall that

$$I_{22}(\alpha,\beta) = I'_{22}(\alpha,\beta) + T^{-\alpha-\beta} I'_{22}(-\beta,-\alpha) + O(T/L),$$

and that

$$I'_{22}(\alpha,\beta) = I'_{220}(\alpha,\beta) + O(T^{1-\varepsilon}).$$

Therefore

$$\begin{aligned} I_{22}(\alpha,\beta) &= I'_{220}(\alpha,\beta) + T^{-\alpha-\beta} I'_{220}(-\beta,-\alpha) + O(T/L) \\ &= (I'_{220}(\alpha,\beta) + I'_{220}(-\beta,-\alpha)) + (T^{-\alpha-\beta}-1) I'_{220}(-\beta,-\alpha) + O(T/L). \end{aligned}$$

We first take a look at the first term in the brackets

$$\frac{d^2}{dxdy}\Bigg[(y_2^{\alpha x+\beta y} - y_2^{-\beta x-\alpha y}) \int_0^1 (1-u)^{\ell_1+\ell_2} P_{\ell_1}(x+u)P_{\ell_2}(y+u)du\Big|_{x=y=0}\Bigg] \\ = (\alpha+\beta)\log y_2 \Bigg(\int_0^1 (1-u)^{\ell_1+\ell_2} P'_{\ell_1}(u)P_{\ell_2}(u)du + \int_0^1 (1-u)^{\ell_1+\ell_2} P_{\ell_1}(u)P'_{\ell_2}(u)du \Bigg).$$

Since $P_{\ell_1}(0) = P_{\ell_2}(0) = 0$, we also have

$$0 = (1-u)^{\ell_1+\ell_2} P_{\ell_1}(u)P_{\ell_2}(u)\Big|_{u=0}^{1} = \int_0^1 \Big((1-u)^{\ell_1+\ell_2} P_{\ell_1}(u)P_{\ell_2}(u)\Big)' du.$$

This implies that

$$(\ell_1+\ell_2)\int_0^1 (1-u)^{\ell_1+\ell_2-1} P_{\ell_1}(u)P_{\ell_2}(u)du \\ = \int_0^1 (1-u)^{\ell_1+\ell_2} P'_{\ell_1}(u)P_{\ell_2}(u)du + \int_0^1 (1-u)^{\ell_1+\ell_2} P_{\ell_1}(u)P'_{\ell_2}(u)du.$$

Combining these observations gives

$$I'_{220}(\alpha,\beta) + I'_{220}(-\beta,-\alpha) = \widehat{w}(0) \sum_{\ell_1=2}^{K}\sum_{\ell_2=2}^{K}\sum_{k=0}^{\min(\ell_1,\ell_2)} (-1)^{\ell_1+\ell_2-2k}\binom{\ell_1}{k}(\ell_2)_k \\ \times \frac{2^{\ell_1+\ell_2-2k}}{(\ell_1+\ell_2-1)!}\int_0^1 (1-u)^{\ell_1+\ell_2-1} P_{\ell_1}(u)P_{\ell_2}(u)du.$$

For the expression $(T^{-\alpha-\beta}-1)I'_{22}(-\beta,-\alpha)$, we again use (6.18) to obtain

$$\frac{\widehat{w}(0)}{\theta_2} \sum_{\ell_1=2}^{K}\sum_{\ell_2=2}^{K}\sum_{k=0}^{\min(\ell_1,\ell_2)} \binom{\ell_1}{k}(\ell_2)_k(-1)^{\ell_1+\ell_2-2k} \frac{2^{\ell_1+\ell_2-2k}}{(\ell_1+\ell_2)!}$$

$$\times \frac{d^2}{dxdy}\left[y_2^{-\beta x-\alpha y}\int_0^1\int_0^1 T^{-v(\alpha+\beta)}(1-u)^{\ell_1+\ell_2}P_{\ell_1}(x+u)P_{\ell_2}(y+u)dudv\bigg|_{x=y=0}\right]$$
$$+O(T/L).$$

By applying similar arguments for the holomorphy of the error terms as in the Section 6.3.1, this completes the proof of Lemma 6.1.3.

Appendix

Special functions and transforms

A.1 Special functions

In this brief appendix, we will introduce the notation and properties of the functions we widely use in this thesis. The most common functions are described in detail in [Tit86], [Edw74], [Tit48], [Gol06], [Bru25] and [IK04].

A.1.1 The Gamma function

The Gamma function is defined for $\mathrm{Re}(s) > 0$ with the following improper integral

$$\Gamma(s) = \int_0^{+\infty} e^{-x} x^{s-1} dx. \tag{A.1}$$

One can simply notice that $\Gamma(1) = 1$ and that using integration by parts,

$$\Gamma(s+1) = s\Gamma(s). \tag{A.2}$$

The integral can be thus extended by analytic continuation to the entire complex plane using (A.2), with the exception of simple poles at $s = -n$ for each $n \in \mathbb{N}_{\geq 0}$ with corresponding residues $\frac{(-1)^n}{n!}$.

Using (A.2), we have for a positive integer $n \in \mathbb{N}$,

$$\Gamma(n+1) = n!. \tag{A.3}$$

Another functional equation is the reflection formula

$$\Gamma(s)\Gamma(1-s) = \frac{\pi}{\sin(\pi s)}, \tag{A.4}$$

due to Euler. One can define the Gamma function with respect to suitable normalizations,

$$\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right), \quad \Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma\left(s\right). \tag{A.5}$$

The duplication formula

$$\Gamma(s)\Gamma\left(s+\frac{1}{2}\right) = 2^{1-2s}\sqrt{\pi}\Gamma(2s), \tag{A.6}$$

holds, and in particular, this implies that

$$\Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1) = \Gamma_{\mathbb{C}}(s). \tag{A.7}$$

Its derivative can be described in terms of the digamma function

$$\Gamma'(s) = \Gamma(s)\psi(s), \tag{A.8}$$

where

$$\psi(s) = \frac{\Gamma'}{\Gamma}(s) = \frac{\Gamma'(s)}{\Gamma(s)} \tag{A.9}$$

The digamma function can be computed more explicitly,

$$\psi(s) = -\gamma + \sum_{n=1}^{\infty} \left(\frac{1}{n+1} - \frac{1}{n+s} \right), \tag{A.10}$$

for $s \neq 0, -1, -2, \ldots$, where γ is the Euler-Mascheroni constant.

Some of its special values are

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}, \quad \Gamma'(1) = -\gamma \tag{A.11}$$

Stirling's asymptotic formula for $\Gamma(s)$ with $s = \sigma + it$ in a vertical bounded strip $\sigma_1 \leq \sigma \leq \sigma_2$ is

$$|\Gamma(\sigma + it)| = \sqrt{2\pi}|t|^{\sigma - \frac{1}{2}} e^{-\frac{1}{2}\pi|t|} \left(1 + O\left(\frac{1}{|t|}\right)\right), \tag{A.12}$$

as $|t| \to \infty$. The formula can also be written as

$$\Gamma(s) = \sqrt{2\pi} e^{-s} s^{s-\frac{1}{2}} \exp(O(|s|^{-1})) \tag{A.13}$$

as $|t| \to \infty$.

A.1.2 The Bessel functions J, Y and K

The Bessel functions of the first kind $J_\nu(x)$ of order ν are defined as the solutions of the following linear differential equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \nu^2)y = 0, \tag{A.14}$$

which is called Bessel equation. They can be expressed as the following absolutely convergent series for each $x \in \mathbb{R}$,

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{\nu+2n}}{n!\Gamma(\nu+n+1)}. \tag{A.15}$$

The Bessel function of the second kind $Y_\nu(x)$ is defined as

$$Y_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)}, \tag{A.16}$$

for non-integers ν. In the case when $\nu = n$ is an integer, then the function $Y_n(x)$ is defined by taking the limit

$$Y_n(x) = \lim_{\nu \to n} Y_\nu(x).$$

The modified Bessel function of the first kind is defined by

$$I_\nu(x) = i^{-\nu} J_\nu(ix) = \sum_{m=0}^{\infty} \frac{(x/2)^{\nu+2m}}{m!\Gamma(\nu+m+1)}. \tag{A.17}$$

The Macdonald-Bessel function $K_\nu(x)$ of order ν, also called the K-Bessel function or modified Bessel function of the second kind is defined as

$$K_\nu(x) = \frac{1}{2}\int_0^\infty e^{-x(t+t^{-1})} t^{\nu-1} dt. \tag{A.18}$$

If $x > 0$, the integrand in the definition decays rapidly as $t \to 0$ and $t \to \infty$. The integral is thus convergent for every ν.

It can be shown that $K_\nu(x)$ is a solution of the modified Bessel equation

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - (x^2 + \nu^2)y = 0. \tag{A.19}$$

It can thus be expressed as a linear combination of modified Bessel's functions of the first kind $I_\nu(x)$,

$$K_\nu(x) = \frac{\pi}{2} \frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\pi\nu)}. \tag{A.20}$$

Moreover, because of (A.20), $K_\nu(x)$ is even in ν,

$$K_\nu(x) = K_{-\nu}(x). \tag{A.21}$$

A.1.3 The hypergeometric functions

The generalized hypergeometric function is written as

$${}_pF_q\begin{pmatrix} a_1 & \dots & a_p \\ b_1 & \dots & b_q \end{pmatrix}; z\Bigg) = \sum_{n=0}^{\infty} \frac{(a_1)_n \dots (a_p)_n}{(b_1)_n \dots (b_q)_n} \frac{z^n}{n!}, \tag{A.22}$$

where

$$(q)_n = \begin{cases} 1, & \text{if } n = 0, \\ q(q+1)\cdots(q+n+1), & \text{if } n > 0, \end{cases} \tag{A.23}$$

is the Pochhammer symbol. When all the terms of the series are defined and it has a non-zero radius of convergence, then the series defines an analytic function.

There are two special cases of particular interest, the Gaussian hypergeometric function ${}_2F_1$ and the confluent hypergeometric function ${}_1F_1$.

The Gaussian hypergeometric function ${}_2F_1$ is defined for $|z| < 1$ as

$${}_2F_1(a, b, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}. \tag{A.24}$$

An important property of ${}_2F_1$ is due to Gauss, which states that

$${}_2F_1(a, b, c; 1) = \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}, \tag{A.25}$$

when $\mathrm{Re}(c) > \mathrm{Re}(a + b)$.

The confluent hypergeometric function ${}_1F_1$ is the hypergeometric series given by

$$ {}_1F_1(a,b;z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n} \frac{z^n}{n!}. \tag{A.26}$$

for $|z| < \infty$. It can be expressed as an integral in the following way,

$$ {}_1F_1(a,b;z) = \int_0^1 e^{zt} t^{a-1} (1-t)^{b-a-1} dt. \tag{A.27}$$

The confluent hypergeometric limit function ${}_0F_1$ is defined similarly, and it is related to the J-Bessel functions (A.15) by the formula

$$ J_\nu(x) = \frac{\left(\frac{x}{2}\right)^\nu}{\Gamma(\nu+1)} {}_0F_1(\nu+1, -\frac{1}{4}x^2). \tag{A.28}$$

A.2 Integral transforms

A.2.1 Fourier transform

The Fourier transform is an operator $\mathcal{F} : L^1(\mathbb{R}) \to L^\infty(\mathbb{R})$ given by the following formula

$$ \mathcal{F}(f)(\xi) = \hat{f}(\xi) := \int_{-\infty}^{\infty} f(x) e^{-2\pi i x \xi} dx \tag{A.29}$$

The domain of the original function is commonly referred as the *time domain*, while its image is called the *frequency domain*.

The trivial inequality,

$$ |\hat{f}(\xi)| \le \int_{-\infty}^{\infty} |f(x)| dx = ||f||_{L^1(\mathbb{R})} \tag{A.30}$$

shows that the Fourier transform is a bounded operator, with corresponding operator norm bounded by 1.

It is not generally possible to write the inverse as a Lebesgue integral. However, when both f and $\hat{f}$ are integrable, the inverse equality

$$ f(x) = \int_{-\infty}^{\infty} \hat{f}(\xi) e^{2\pi i x \xi} d\xi \tag{A.31}$$

holds almost everywhere.

Basic properties of the Fourier transform are listed below.

1. *Linearity*. For $a, b \in \mathbb{C}$, if $h(x) = af(x) + bg(x)$, then

$$ \hat{h}(\xi) = a\hat{f}(\xi) + b\hat{g}(\xi). \tag{A.32}$$

2. *Translation/time shifting*. For any $x_0 \in \mathbb{R}$, if $h(x) = f(x - x_0)$, then

$$ \hat{h}(\xi) = e^{-2\pi i x_0 \xi} \hat{f}(\xi). \tag{A.33}$$

3. *Modulation/frequancy shifting*. For any $\xi_0 \in \mathbb{R}$, if $h(x) = e^{2\pi i x \xi_0} f(x)$, then
$$\hat{h}(\xi) = \hat{f}(\xi - \xi_0). \tag{A.34}$$

4. *Time scaling*. For $a \in \mathbb{R}$ non-zero, and if $h(x) = f(ax)$, then
$$\hat{h}(\xi) = \frac{1}{|a|}\hat{f}\left(\frac{\xi}{a}\right). \tag{A.35}$$

5. *Conjugation*. If $h(x) = \overline{f(x)}$, then
$$\hat{h}(\xi) = \overline{\hat{f}(-\xi)}. \tag{A.36}$$
In particular, if f is real and even, then its Fourier transform is also real and even.

6. *Riemann-Lebesgue Lemma*. If $f \in L^1(\mathbb{R})$, then
$$\hat{f}(\xi) \to 0 \tag{A.37}$$
as $|\xi| \to \infty$.

The Plancherel theorem states that the Fourier transform is a unitary operator on $L^2(\mathbb{R})$, meaning that
$$\int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{f}(\xi)|^2 d\xi \tag{A.38}$$
This implies that the Fourier transform map restricted to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ has a unique extension to a linear isometric map $L^2(\mathbb{R}) \to L^2(\mathbb{R})$.

For a sufficient regular function f, the Poisson summation formula holds,
$$\sum_{n=-\infty}^{\infty} f(n) = \sum_{k=-\infty}^{\infty} \hat{f}(k). \tag{A.39}$$

A.2.2 Mellin transform

The Mellin transform of a function f is an integral transform defined as
$$\mathcal{M}(f)(s) = F(s) := \int_0^{\infty} f(x) x^{s-1} dx. \tag{A.40}$$
If $F(s)$ is analytic in the strip $a < \mathrm{Re}(s) < b$ and it tends to zero uniformly as $|\mathrm{Im}(s)| \to \infty$, then for any c between a and b the Mellin inversion theorem
$$\mathcal{M}^{-1}(F)(x) := \frac{1}{2\pi i}\int_{(c)} \varphi(s) x^{-s} ds = f(x) \tag{A.41}$$
holds.

The Mellin transform is connected to the Fourier transform (A.29) by the following identity
$$\mathcal{M}(f)(s) = \mathcal{F}(f(e^{-x}))(-is).$$

The most common example of a Mellin transfom is the Γ-function (A.1), which can be expressed as

$$\mathcal{M}(e^{-x})(s) := \Gamma(s) = \int_0^{+\infty} e^{-x}x^{s-1}dx. \tag{A.42}$$

Basic properties of the Mellin transform are listed below.

1. *Linearity*. For $a, b \in \mathbb{C}$, if $h(x) = af(x) + bg(x)$ with $F(s)$, $G(s)$ the Mellin transforms of $f(x)$ and $g(x)$ respectively, then the Mellin transform $H(s)$ of $h(x)$ is given by
$$H(s) = aF(s) + bG(s). \tag{A.43}$$

2. *Scaling property*. For any $a \in \mathbb{C}$,
$$\mathcal{M}(f(ax))(s) = a^{-s}F(s). \tag{A.44}$$

3. *Multiplication by x^a*. For any $a \in \mathbb{C}$,
$$\mathcal{M}(x^a f(x))(s) = F(s+a). \tag{A.45}$$

4. *Multiplication by* $\log x$. We have
$$\mathcal{M}(\log x f(x))(s) = \frac{d}{ds}F(s). \tag{A.46}$$

5. *Derivative*. For any $k \geq 1$, we have
$$\mathcal{M}\left(\frac{d^k}{dx^k}f(x)\right)(s) = (-1)^k(s-k)_k F(s), \tag{A.47}$$
where $(s)_k$ is the Pochhammer symbol (A.23).

A.3 Holomorphic modular forms

Classical modular forms are special holomorphic functions defined on the upper half-plane that satisfy the modularity property (A.54) and have a moderate growth at infinity. They have various applications in number theory.

The group

$$\Gamma(1) := SL(2,\mathbb{Z}) = \left\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \;:\; a,b,c,d \in \mathbb{Z}, ad-bc = 1\right\} \tag{A.48}$$

is called *modular group*. It acts on the *upper half-plane*

$$\mathbb{H} := \{z = x+iy \in \mathbb{C} \;:\; y > 0\} \tag{A.49}$$

by fractional linear transformation

$$\gamma \cdot z := \frac{az+b}{cz+d}, \tag{A.50}$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $z \in \mathbb{H}$.

The orbit set $\Gamma \setminus \mathbb{H}$ is a noncompact Riemann surface. Note that $\mathbb{H}$ can be realized by $SL(2,\mathbb{R})/SO(2,\mathbb{R})$ via the following map using the orbit-stabilizer theorem

$$\begin{aligned} SL(2,\mathbb{R})/SO(2,\mathbb{R}) &\to \mathbb{H} \\ \gamma SO(2,\mathbb{R}) &\mapsto \gamma \cdot i \end{aligned} \tag{A.51}$$

since $SO(2,\mathbb{R})$ is the stabilizer of $i \in \mathbb{H}$ and the group action is transitive.

The fundamental domain for the action of $\Gamma(1)$ on $\mathbb{H}$ is

$$\mathcal{F} := \left\{ z \in \mathbb{H} \ : \ -\frac{1}{2} < \operatorname{Re}(z) < \frac{1}{2}, \ |z| > 1 \right\} \tag{A.52}$$

The group $\Gamma(1)$ is generated by

$$\Gamma(1)/\{\pm 1\} = < T, S > \tag{A.53}$$

where $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ and $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, which means that every element of $\Gamma(1)$ can be expressed as a finite combination of T, S, T^{-1}, S^{-1}.

A *modular form* of weight k is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ such that

$$(f|_k\gamma)(z) := (cz+d)^{-k} f(\gamma \cdot z) = f(z) \tag{A.54}$$

for all $\gamma \in \Gamma(1)$, and it's holomorphic at ∞, meaning that its Fourier series is a Taylor series in $q = \exp(2\pi i z)$,

$$f(q) = \sum_{n \geq 0} b(n) q^n. \tag{A.55}$$

The space of modular forms of weight k is denoted by $M_k(\Gamma(1))$.

A modular form of weight k is called *cusp form* if if it vanished at infinity, i.e. satisfying

$$\lim_{y \to +\infty} f(z) = b(0) = 0.$$

The subspace of cusp forms of weight k is denoted by $S_k(\Gamma(1))$.

The Hecke operators are defined as

$$T_n f(z) = \frac{1}{\sqrt{n}} \sum_{\substack{ad=n \\ 0 \leq b < d}} f\left(\frac{az+b}{d}\right). \tag{A.56}$$

The Hecke operators satisfy, for any $m, n \geq 1$,

$$T_m T_n = \sum_{d|(n,m)} d^{k-1} T_{mn/d^2}. \tag{A.57}$$

In particular, the Hecke operators commute: $T_m T_n = T_n T_m$. From the same formula, one can easily check that $T_m T_n = T_{mn}$ if $(m,n) = 1$. Moreover, for every prime p and every integer $m \geq 1$,

$$T_{p^m} T_p = T_{p^{m+1}} + p^{k-1} T_{p^{m-1}}.$$

We say that a modular form is a *Hecke eigenform* if it is an eigenvalue of all Hecke operators for all $n \geq 1$.

Let $f(z)$ be a cusp form

$$f(z) = \sum_{n=1}^{\infty} b(n) q^n$$

which is a Hecke eigenform and it is normalized such that $b(1) = 1$. Then we define its L-function $L(s, f)$ by

$$L(s, f) = \sum_{n=1}^{\infty} \frac{b(n)}{n^s}. \tag{A.58}$$

Then $L(s, f)$ continues to an entire L-function with functional equation

$$\Lambda(s) := (2\pi)^{-s} \Gamma(s) L(s, f) = (-1)^{k/2} \Lambda(k - s), \tag{A.59}$$

and has an Euler-product of the form

$$L(s, f) = \prod_p \left(1 - b(p) p^{-s} + p^{k-1} p^{-2s}\right)^{-1} \tag{A.60}$$

for $\mathrm{Re}(s) > \frac{k}{2} + 1$.

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